

UNIFORM REGULARITY AND VANISHING VISCOSITY LIMIT FOR THE COMPRESSIBLE NEMATIC LIQUID CRYSTAL FLOWS IN THREE DIMENSIONAL BOUNDED DOMAIN

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ABSTRACT. In this paper, we study the uniform regularity and vanishing viscosity limit for the compressible nematic liquid crystal flows in three dimensional bounded domain. It is shown that there exists a unique strong solution for the compressible nematic liquid crystal flows with boundary condition in a finite time interval which is independent of the viscosity coefficient. The solutions are uniform bounded in a conormal Sobolev space. Furthermore, we prove that the density and velocity are uniform bounded in $W^{1,\infty}$, and the director field is uniform bounded in $W^{3,\infty}$ respectively. Based on these uniform estimates, one also obtains the convergence rate of the viscous solutions to the inviscid ones with a rate of convergence.

2010 Mathematics Subject Classification: 35Q35, 35B65, 76A15.

Keywords: Nematic liquid crystal flows, Vanishing viscosity limit, Convergence rate, Conormal Sobolev space.

1. INTRODUCTION

In this paper, we investigate the motion of compressible nematic liquid crystal flows, which are governed by the following simplified version of the Ericksen-Leslie equations as follows

$$\begin{cases} \rho_t^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, & (x, t) \in \Omega \times (0, T), \\ \rho^\varepsilon u_t^\varepsilon + \rho^\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon = \mu \varepsilon \Delta u^\varepsilon + (\mu + \lambda) \varepsilon \nabla \operatorname{div} u^\varepsilon - \nabla d^\varepsilon \cdot \Delta d^\varepsilon, & (x, t) \in \Omega \times (0, T), \\ d_t^\varepsilon + u^\varepsilon \cdot \nabla d^\varepsilon = \Delta d^\varepsilon + |\nabla d^\varepsilon|^2 d^\varepsilon, & (x, t) \in \Omega \times (0, T). \end{cases} \quad (1.1)$$

Here $0 < T \leq +\infty$ and Ω is a bounded smooth domain of \mathbb{R}^3 . The unknown functions $\rho^\varepsilon(x, t)$, $u^\varepsilon(x, t) = (u_1^\varepsilon(x, t), u_2^\varepsilon(x, t), u_3^\varepsilon(x, t),)$ and $d^\varepsilon(x, t) = (d_1^\varepsilon(x, t), d_2^\varepsilon(x, t), d_3^\varepsilon(x, t))$ represent the density, velocity field of fluid and the macroscopic average of the nematic liquid crystal orientation field respectively. The scalar function $p^\varepsilon = p(\rho^\varepsilon)$ is the pressure function given by γ -law

$$p(\rho) = \rho^\gamma \quad \text{with} \quad \gamma > 1.$$

The viscous coefficients μ and λ satisfy the physical restrictions

$$\mu > 0, \quad 2\mu + 3\lambda > 0,$$

where the parameter $\varepsilon > 0$ is the inverse of the Reynolds number. For more results about the compressible Ericksen-Leslie system (1.1), the readers can refer to [1, 2, 3, 4, 5, 6, 7, 8] and references therein. Corresponding to the system (1.1), we impose the following Navier-slip type and Neumann boundary conditions:

$$u^\varepsilon \cdot n = 0, \quad ((Su^\varepsilon)n)_\tau = -(Au^\varepsilon)_\tau, \quad \text{and} \quad \frac{\partial d^\varepsilon}{\partial n} = 0, \quad \text{on } \partial\Omega, \quad (1.2)$$

where A is a given smooth symmetric matrix(see [9]), n is the outward unit vector normal to $\partial\Omega$, $(Au^\varepsilon)_\tau$ represents the tangential component of Au^ε . The strain tensor Su^ε is defined by

$$Su^\varepsilon = \frac{1}{2} ((\nabla u^\varepsilon) + (\nabla u^\varepsilon)^t).$$

For any smooth solutions v , it is easy to check that

$$(2S(v)n - (\nabla \times v) \times n)_\tau = -(2S(n)v)_\tau,$$

see [10] for detail. Then the boundary condition (1.2) can be written in the form of the vorticity as

$$u^\varepsilon \cdot n = 0, \quad n \times (\nabla \times u^\varepsilon) = [Bu^\varepsilon]_\tau, \quad \text{and} \quad \frac{\partial d^\varepsilon}{\partial n} = 0, \quad \text{on } \partial\Omega, \quad (1.3)$$

where $B = 2(A - S(n))$ is symmetric matrix. Actually, it turns out that the form (1.3) will be more convenient than (1.2) in the energy estimates, see [11, 12].

In this paper, we are interested in the existence of strong solution of (1.1) with uniform bounds on an interval of time independent of viscosity $\varepsilon \in (0, 1]$ and the vanishing viscosity limit to the corresponding invicid nematic liquid crystal flows as ε vanishes, i.e.

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, & (x, t) \in \Omega \times (0, T), \\ \rho u_t + \rho u \cdot \nabla u + \nabla p = -\nabla d \cdot \Delta d, & (x, t) \in \Omega \times (0, T), \\ d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, & (x, t) \in \Omega \times (0, T). \end{cases} \quad (1.4)$$

with the boundary condition

$$u \cdot n = 0, \quad \text{and} \quad \frac{\partial d}{\partial n} = 0, \quad \text{on } \partial\Omega. \quad (1.5)$$

When the density and director field are constant scalar function and constant vector field respectively, the systems (1.1) and (1.4) are the well-known incompressible Navier-Stokes equations and incompressible Euler equations respectively. There is lots of literature on the uniform bounds and the vanishing viscosity limit for the Navier-Stokes equations without boundaries, for instance, [13, 14, 15, 16]. The appearance of boundary gives arise to the time of existence T^ε depending on the viscosity coefficient, and it is difficult to prove that it stays bounded away from zero. Nevertheless, in a domain with boundaries, for some special types of Navier-slip boundary conditions, some uniform H^3 (or $W^{2,p}$, with p large enough) estimates and a uniform time of existence for Navier-Stokes when the viscosity goes to zero have recently been obtained (see [17, 18, 11]). It is easy to see that, for these special boundary conditions, the main part of the boundary layer vanishes, which allows this uniform control in some limited regularity Sobolev space. Recently, Masmoudi and Rousset [19] established conormal uniform estimates for three-dimensional general smooth domains with the Navier-slip boundary condition and obtained convergence of the viscous solutions to the inviscid ones by a compact argument. Based on the uniform estimates in [9], better convergence with rates have been studied in [9] and [20]. In particular, Xiao and Xin [20] have proved the convergence in $L^\infty(0, T; H^1)$ with a rate of convergence. Motivated by the work of [19] and Xin [20], We [21] investigated the vanishing viscosity limit of incompressible nematic liquid crystal flows. More precisely, we proved that there exists a unique strong solution for the incompressible nematic liquid crystal flows in a finite time interval which is independent of the viscosity coefficient and obtained the convergence rate of the viscous solutions to the inviscid ones with a rate of convergence.

For the compressible Navier-Stokes equations, Paddick [22] obtained uniform estimates for the solutions of the compressible isentropic Navier-Stokes equations in the 3-D half-space with a Navier boundary condition, which was improved by Wang et al. [12] to generalized bounded domain. Specially, Wang et al. [12] shown that the boundary layers for the density must be weaker than the one for the velocity and established the convergence of the viscous solutions to the inviscid ones. For more results about the inviscid limit for the compressible Navier-Stokes equations, the readers can refer to [23, 24] and the references therein. Motivated the work of [21] and [12], we hope the investigate the vanishing viscosity limit for the compressible nematic liquid crystal flows (1.1).

Before stating our main results, we first explain the notations and conventions used throughout this paper. Similar to [19, 12], one assumes that the bounded domain $\Omega \subset \mathbb{R}^3$ has a covering that

$$\Omega \subset \Omega_0 \cup_{k=1}^n \Omega_k, \quad (1.6)$$

where $\overline{\Omega}_0$, and in each Ω_k there exists a function ψ_k such that

$$\begin{aligned} \Omega \cap \Omega_k &= \{x = (x_1, x_2, x_3) | x_3 > \psi_k(x_1, x_2)\} \cap \Omega_k, \\ \partial\Omega \cap \Omega_k &= \{x_3 = \psi_k(x_1, x_2)\} \cap \Omega_k. \end{aligned}$$

Here, Ω is said to be \mathcal{C}^m if the functions ψ_k are a \mathcal{C}^m -function. To define the conormal Sobolev spaces, one considers $(Z_k)_{1 \leq k \leq N}$ to be a finite set of generators of vector fields that are tangential to $\partial\Omega$, and sets

$$H_{co}^m = \{f \in L^2(\Omega) | Z^I f \in L^2(\Omega), \text{ for } |I| \leq m\},$$

where $I = (k_1, \dots, k_m)$. The following notations will be used

$$\begin{aligned} \|u\|_m^2 &= \|u\|_{H_{co}^m}^2 = \sum_{j=1}^3 \sum_{|I| \leq m} \|Z^I u_j\|_{L^2}^2, \\ \|u\|_{m,\infty}^2 &= \sum_{|I| \leq m} \|Z^I u\|_{L^\infty}^2, \end{aligned}$$

and

$$\|\nabla Z^m u\|^2 = \sum_{|I|=m} \|\nabla Z^I u\|_{L^2}^2.$$

Noting that by using the covering of Ω , one can always assume that each vector field $(p^\varepsilon, u^\varepsilon, d^\varepsilon)$ is supported in one of the Ω_i , and moreover, in Ω_0 the norm $\|\cdot\|_m$ yields a control of the standard H^m norm, whereas if $\Omega_i \cap \partial\Omega \neq \emptyset$, there is no control of the normal derivatives.

Since $\partial\Omega$ is given locally by $x_3 = \psi(x_1, x_2)$ (we omit the subscript j of notational convenience), it is convenient to use the coordinates

$$\Psi : (y, z) \mapsto (y, \psi(y) + z) = x.$$

A basis is thus given by the vector fields (e_{y^1}, e_{y^2}, e_z) , where $e_{y^1} = (1, 0, \partial_1 \psi)^t$, $e_{y^2} = (0, 1, \partial_2 \psi)^t$, and $e_z = (0, 0, -1)^t$. On the boundary, e_{y^1} and e_{y^2} are tangent to $\partial\Omega$, and in general, e_z is not a normal vector field. By using this parametrization, one can take as suitable vector fields compactly supported in Ω_j in the definition of the $\|\cdot\|_m$ norms

$$Z_i = \partial_{y^i} = \partial_i + \partial_i \psi \partial_z, \quad i = 1, 2, \quad Z_3 = \varphi(z) \partial_z,$$

where $\varphi(z) = \frac{z}{1+z}$ is smooth, supported in \mathbb{R}_+ with the property $\varphi(0) = 0, \varphi'(0) > 0, \varphi(z) > 0$ for $z > 0$. It is easy to check that

$$Z_k Z_j = Z_j Z_k, \quad j, k = 1, 2, 3, \quad (1.7)$$

and

$$\partial_z Z_i = Z_i \partial_z, \quad i = 1, 2; \quad \partial_z Z_3 \neq Z_3 \partial_z.$$

We shall still denote by ∂_j , $j = 1, 2, 3$, or ∇ the derivatives in the physical space. The coordinates of a vector field u in the basis (e_{y^1}, e_{y^2}, e_z) will be denoted by u^i , and thus

$$u = u^1 e_{y^1} + u^2 e_{y^2} + u^3 e_z.$$

We shall denote by u_j the coordinates in the standard basis of \mathbb{R}^3 , i.e. $u = u_1 e_1 + u_2 e_2 + u_3 e_3$. Denote by n the unit outward normal in the physical space which is given locally by

$$n(x) \equiv n(\Psi(y, z)) = \frac{1}{\sqrt{1 + |\nabla \psi(y)|^2}} \begin{pmatrix} \partial_1 \psi(y) \\ \partial_2 \psi(y) \\ -1 \end{pmatrix} \triangleq \frac{-N(y)}{\sqrt{1 + |\nabla \psi(y)|^2}}, \quad (1.8)$$

and by Π the orthogonal projection

$$\Pi u \equiv \Pi(\Psi(y, z))u = u - [u \cdot n(\Psi(y, z))]n(\Psi(y, z)), \quad (1.9)$$

which gives the orthogonal projection on to the tangent space of the boundary. Note that n and Π are defined in the whole Ω_k and do not depend on z . For later use and notational convenience, set

$$\mathcal{Z}^\alpha = \partial_t^{\alpha_0} Z^{\alpha_1} = \partial_t^{\alpha_0} Z_1^{\alpha_{11}} Z_2^{\alpha_{12}} Z_3^{\alpha_{13}},$$

where α, α_0 and α_1 are the differential multi-indices with $\alpha = (\alpha_0, \alpha_1), \alpha_1 = (\alpha_{11}, \alpha_{12}, \alpha_{13})$ and we also use the notation

$$\|f(t)\|_{\mathcal{H}^m}^2 = \|f(t)\|_{\mathcal{H}^m}^2 = \sum_{|\alpha| \leq m} \|\mathcal{Z}^\alpha f(t)\|_{L_x^2}^2, \quad (1.10)$$

and

$$\|f(t)\|_{\mathcal{H}^{k,\infty}} = \sum_{|\alpha| \leq k} \|\mathcal{Z}^\alpha f(t)\|_{L_x^\infty} \quad (1.11)$$

for smooth space-time function $f(x, t)$. Throughout this paper, the positive generic constants that are independent of ε are denoted by c, C . Denote by C_k a positive constant independent of $\varepsilon \in (0, 1]$ which depends only on the \mathcal{C}^k -norm of the functions ψ_j , $j = 1, \dots, n$. Here, $\|\cdot\|_{L^2}$ denotes the standard $L^2(\Omega; dx)$ norm, and $\|\cdot\|_{H^m}$ ($m = 1, 2, 3, \dots$) denotes the Sobolev $H^m(\Omega, dx)$ norm. The notation $|\cdot|_{H^m}$ will be used for the standard Sobolev norm of functions defined on $\partial\Omega$. Note that this norm involves only tangential derivatives. $P(\cdot)$ denotes a polynomial function.

Since the boundary layers may appear in the presence of physical boundaries, in order to obtain the uniform estimates for solutions to the nematic liquid crystal flows with Navier-slip and Neumann boundary conditions, we need to find a suitable functional space. In the spirit of Wang et al. [12], we also investigate the solutions of the nematic liquid crystal flows in Conormal Sobolev space. Hence, the functional space should include some information for the direction field d . On the other hand, due to the nonlinear higher order derivatives term $\nabla d \cdot \Delta d$, one should control this term by using the dissipative term Δd on the right hand side of the equation (1.1)₂ which involving the time derivatives term d_t . Hence, we also include some information involving the time derivatives in the functional space. Therefore, we define the functional space $X_m^\varepsilon(T)$ for a pair of function $(u, p, d)(x, t)$ as follows

$$X_m^\varepsilon(T) = \{(p, u, d) \in L^\infty([0, T], L^2); \text{esssup}_{0 \leq t \leq T} \|(p, u, d)(t)\|_{X_m^\varepsilon} < +\infty\}, \quad (1.12)$$

where the norm $\|(\cdot, \cdot)\|_{X_m^\varepsilon}$ is given by

$$\begin{aligned} \|(p, u, d)(t)\|_{X_m^\varepsilon} &= \|(u, p)(t)\|_{\mathcal{H}^m}^2 + \|d(t)\|_{L^2}^2 + \|\nabla d(t)\|_{\mathcal{H}^m}^2 + \|(\nabla u, \Delta d)(t)\|_{\mathcal{H}^{m-1}}^2 \\ &\quad + \|\nabla u(t)\|_{\mathcal{H}^{1,\infty}}^2 + \sum_{k=0}^{m-2} \|\partial_t^k \nabla p(t)\|_{\mathcal{H}^{m-1-k}}^2 + \varepsilon \|\nabla \partial_t^{m-1} p(t)\|_{L^2}^2 \\ &\quad + \|\Delta p(t)\|_{\mathcal{H}^1}^2 + \varepsilon \|\Delta p(t)\|_{\mathcal{H}^2}^2. \end{aligned} \quad (1.13)$$

In the present paper, we supplement the nematic liquid crystal flows system (1.1) with initial data

$$(p^\varepsilon, u^\varepsilon, d^\varepsilon)(x, 0) = (p_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon)(x), \quad (1.14)$$

such that

$$0 < \frac{1}{\hat{C}_0} \leq \rho_0^\varepsilon \leq \hat{C}_0 < \infty, \quad (1.15)$$

and

$$\begin{aligned} &\sup_{0 < \varepsilon \leq 1} \|(p_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon)\|_{X_m^\varepsilon} \\ &= \sup_{0 < \varepsilon \leq 1} \{ \|(u_0^\varepsilon, p_0^\varepsilon)\|_{\mathcal{H}^m}^2 + \|d_0^\varepsilon\|_{L^2}^2 + \|\nabla d_0^\varepsilon\|_{\mathcal{H}^m}^2 + \|\nabla u_0^\varepsilon\|_{\mathcal{H}^{m-1}}^2 + \sum_{k=0}^{m-2} \|\partial_t^k \nabla p_0^\varepsilon\|_{\mathcal{H}^{m-1-k}}^2 \\ &\quad + \varepsilon \|\nabla \partial_t^{m-1} p_0^\varepsilon\|_{L^2}^2 + \|\Delta p_0^\varepsilon\|_{\mathcal{H}^1}^2 + \varepsilon \|\Delta p_0^\varepsilon\|_{\mathcal{H}^2}^2 + \|\Delta d_0^\varepsilon\|_{\mathcal{H}^{m-1}}^2 + \|\nabla u_0^\varepsilon\|_{\mathcal{H}^{1,\infty}}^2 \} \leq \tilde{C}_0, \end{aligned} \quad (1.16)$$

where \tilde{C}_0 is a positive constant independent of $\varepsilon \in (0, 1]$, and the time derivatives of initial data are defined through the equation (1.1). Thus, the initial data $(\rho_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon)$ is assumed to have a higher space regularity and compatibilities. Notice that the a priori estimates in Theorem 3.1 below are obtained in the case that the approximate solution is sufficiently smooth up to the boundary, and therefore, in order to obtain a selfcontained result, one needs to assume the approximated initial data satisfies the boundary compatibilities condition (1.3). For the initial data $(\rho_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon)$ satisfying (1.14), it is not clear if there exists an approximate sequences $(\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}, d_0^{\varepsilon,\delta})$ (δ being a regularization

parameter) which satisfy the boundary compatibilities and $\|(p_0^{\varepsilon,\delta} - p_0^\varepsilon, u_0^{\varepsilon,\delta} - u_0^\varepsilon, d_0^{\varepsilon,\delta} - d_0^\varepsilon)\|_{X_m^\varepsilon} \rightarrow 0$ as $\delta \rightarrow 0$. Therefore, we set

$$X_{n,m}^{\varepsilon,ap} = \left\{ (p, u, d) \in [H^{4m}(\Omega)]^2 \times H^{4(m+1)}(\Omega) \mid \partial_t^k p, \partial_t^k u, \partial_t^k d, k = 1, \dots, m \text{ are defined} \right. \\ \left. \text{through the equations (1.1) and } \partial_t^k u, \right. \\ \left. \partial_t^k \nabla d, k = 0, \dots, m-1, \text{ satisfy the} \right. \\ \left. \text{boundary compatibility condition} \right\}. \quad (1.17)$$

and

$$X_{n,m}^\varepsilon = \text{the closure of } X_{n,m}^{\varepsilon,ap} \text{ in the norm } \|(\cdot, \cdot)\|_{X_m^\varepsilon}. \quad (1.18)$$

Now, we state the first results concerning the uniform regularity for the nematic liquid crystal flows (1.1), (1.3) and (1.14) as follows.

Theorem 1.1 (Uniform Regularity). *Let m be an integer satisfying $m \geq 6$, Ω be a C^{m+2} domain, and $A \in C^{m+1}(\partial\Omega)$. Consider the initial data $(p_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon) \in X_{n,m}^\varepsilon$ satisfy (1.16) and $|d_0^\varepsilon| = 1$ in $\bar{\Omega}$. Then, there exists a time $T_0 > 0$ and $\tilde{C}_1 > 0$ independent of $\varepsilon \in (0, 1]$, such that there exists a unique solution of (1.1), (1.3) and (1.14) which is defined on $[0, T_0]$ and satisfies the estimates*

$$\begin{aligned} & \sup_{0 \leq t \leq T_0} (\|d^\varepsilon(t)\|_{L^2}^2 + \|(u^\varepsilon, p^\varepsilon, \nabla d^\varepsilon)(t)\|_{\mathcal{H}^m}^2 + \|(\nabla u^\varepsilon, \Delta d^\varepsilon)(t)\|_{\mathcal{H}^{m-1}}^2 + \|\nabla u^\varepsilon(t)\|_{\mathcal{H}^{1,\infty}}^2) \\ & + \sup_{0 \leq t \leq T_0} \left(\sum_{k=0}^{m-2} \|\partial_t^k \nabla p^\varepsilon(t)\|_{m-1-k}^2 + \varepsilon \|\partial_t^{m-1} \nabla p^\varepsilon(t)\|_{L^2}^2 + \|\Delta p^\varepsilon(t)\|_{\mathcal{H}^1}^2 + \varepsilon \|\Delta p^\varepsilon(t)\|_{\mathcal{H}^2}^2 \right) \\ & + \int_0^{T_0} (\|\nabla \partial_t^{m-1} p^\varepsilon(t)\|_{L^2}^2 + \|\Delta p^\varepsilon(t)\|_{\mathcal{H}^2}^2) dt + \varepsilon \int_0^{T_0} \|\nabla u^\varepsilon(t)\|_{\mathcal{H}^m}^2 dt \\ & + \varepsilon^2 \int_0^{T_0} \|\nabla^2 \partial_t^{m-1} u^\varepsilon(t)\|_{L^2}^2 dt + \varepsilon \sum_{k=0}^{m-2} \int_0^{T_0} \|\nabla^2 \partial_t^k u^\varepsilon(t)\|_{m-k-1}^2 dt \\ & + \int_0^{T_0} \|\Delta d^\varepsilon\|_{\mathcal{H}^m}^2 dt + \int_0^{T_0} \|\nabla \Delta d^\varepsilon\|_{\mathcal{H}^{m-1}}^2 dt \leq \tilde{C}_1, \end{aligned} \quad (1.19)$$

and

$$\frac{1}{2\hat{C}_0} \leq \rho^\varepsilon(t) \leq 2\hat{C}_0, \quad t \in [0, T_0], \quad (1.20)$$

where \tilde{C}_1 depends only on \hat{C}_0, \tilde{C}_0 and C_{m+2} .

Remark 1.2. For $(p_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon) \in X_{n,m}^\varepsilon$, it must hold that $u_0^\varepsilon \cdot n|_{\partial\Omega} = 0$, $((Su_0^\varepsilon)n)_\tau|_{\partial\Omega} = -(Au_0^\varepsilon)_\tau|_{\partial\Omega}$, and $n \cdot \nabla d_0^\varepsilon|_{\partial\Omega} = 0$ in the trace sense for every fixed $\varepsilon \in (0, 1]$.

The main steps of the proof of Theorem 1.1 are the following. First, we obtained a conormal energy estimates for $(p^\varepsilon, u^\varepsilon, \nabla d^\varepsilon)$ in \mathcal{H}^m -norm. The second step is to give the estimate for $\|\partial_n u^\varepsilon\|_{\mathcal{H}^{m-1}}$. In order to obtain this estimate by an energy method, $\partial_n u^\varepsilon$ is not a convenient quantity because it does not vanish on the boundary. Similar to Wang et al. [12], $\partial_n u^\varepsilon$ can be controlled by $\partial_n u^\varepsilon \cdot n$ (or $\text{div} u^\varepsilon$) and $(\partial_n u)_\tau$. In order to give the estimate for $(\partial_n u^\varepsilon)_\tau$, one choose the convenient quantity $\eta = w^\varepsilon \times n + (Bu^\varepsilon)_\tau$ with a homogeneous Dirichlet boundary conditions. The third step is to give the estimates for Δd^ε and $\text{div} u^\varepsilon$. Indeed, it is easy to obtain the estimate for the quantity Δd^ε since there exists a dissipative term Δd^ε on the right-hand side of (1.1)₃. In the spirit of Wang et al. [12], we obtain a control of $\sum_{j=0}^{m-2} \|\partial_t^j (\text{div} u^\varepsilon, \nabla p^\varepsilon)\|_{m-1-j}^2$ at the cost that the term $\int_0^t \|\nabla \mathcal{Z}^{m-2} \text{div} u^\varepsilon\|_{L^2}^2 d\tau$ appears in the right-hand side of the inequality. Following the idea as Wang et al. [12], we can obtain the uniform estimates for $\int_0^t \|\partial_t^{m-1} \nabla p^\varepsilon\|_{L^2}^2 d\tau$ and get a control of $\|\partial_t^{m-1} \text{div} u^\varepsilon\|_{L^2}^2$ in terms of $\sum_{j=0}^{m-2} \|\partial_t^j (\nabla u^\varepsilon, \nabla p^\varepsilon)\|_{m-1-j}^2$ and $\|(p^\varepsilon, u^\varepsilon)\|_{\mathcal{H}^m}^2$. The fourth step is to estimate $\|\Delta d^\varepsilon\|_{W^{1,\infty}}$. Indeed, this estimate is easy to obtain since there exists a dissipation term Δd^ε on the right-hand side of (1.1)₃. The fifth step is to estimate $\|\nabla u^\varepsilon\|_{\mathcal{H}^{1,\infty}}$. In fact, it suffices

to estimate $\|(\partial_n u^\varepsilon)_\tau\|_{\mathcal{H}^{1,\infty}}$ since the other terms can be estimated by the Sobolev embedding. We choose an equivalent quantity such that it satisfies a homogeneous Dirichlet condition and solves a convection-diffusion equation at the leading order. The last step is to obtain the uniform estimate of $\|\Delta p^\varepsilon\|_{\mathcal{H}^1}$, which gives a control of $\|\nabla p^\varepsilon\|_{\mathcal{H}^{1,\infty}}$ from Proposition 2.3. Then Theorem 1.1 can be proved by these a priori estimates and a classical iteration method.

Next, we hope to prove the vanishing viscosity limit with rates of convergence, which can be stated as follows.

Theorem 1.3 (Inviscid Limit). *Let $(\rho, u, d)(t) \in L^\infty(0, T_1; H^3 \times H^3 \times H^4)$ be the smooth solution to the equation (1.4) and boundary condition (1.3) with initial data (ρ_0, u_0, d_0) satisfying*

$$(\rho_0, u_0, d_0) \in (H^3 \times H^3 \times H^4) \cap X_{n,m}^\varepsilon \text{ with } m \geq 6. \quad (1.21)$$

Let $(\rho^\varepsilon, u^\varepsilon, d^\varepsilon)(t)$ be the solution to the initial boundary value problem of the nematic liquid crystal flows (1.1), (1.2) with initial data (ρ_0, u_0, d_0) satisfying (1.21). Then, there exists $T_2 = \min\{T_0, T_1\} > 0$, which is independent of $\varepsilon > 0$, such that

$$\|(\rho^\varepsilon - \rho, u^\varepsilon - u)(t)\|_{L^2}^2 + \|(d^\varepsilon - d)(t)\|_{H^1}^2 \leq C\varepsilon^{\frac{3}{2}}, \quad t \in [0, T_2] \quad (1.22)$$

$$\|(\rho^\varepsilon - \rho, u^\varepsilon - u)(t)\|_{H^1}^2 \leq C\varepsilon^{\frac{1}{6}}, \quad \|(d^\varepsilon - d)(t)\|_{H^2}^2 \leq C\varepsilon^{\frac{1}{2}}, \quad t \in [0, T_2] \quad (1.23)$$

and

$$\|(\rho^\varepsilon - \rho, u^\varepsilon - u)\|_{L^\infty(0, T_2; L^\infty(\Omega))} + \|(d^\varepsilon - d)\|_{L^\infty(0, T_2; W^{1,\infty}(\Omega))} \leq C\varepsilon^{\frac{3}{10}}, \quad (1.24)$$

which C depends only on the norm $\|(\rho_0, u_0)\|_{H^3}, \|d_0\|_{H^4}$ and $\|(p(\rho_0), u_0, d_0)\|_{X_{m,n}^\varepsilon}$.

The rest of the paper is organized as follows: In section 2, we collect some inequalities that will be used later. In section 3, the a priori estimates in Theorem 3.1 are proved. By using these a priori estimates, one give the proof for the Theorem 1.1 in section 4. Based on the uniform estimates obtained in Theorem 1.1, we establish the convergence rate for the solutions from (1.1) to (1.4) and complete the proof for Theorem 1.3.

2. PRELIMINARIES

The following lemma [11, 25] allows us to control the $H^m(\Omega)$ -norm of a vector valued function u by its H^{m-1} -norm of $\nabla \times u$ and $\operatorname{div} u$, together with the $H^{m-\frac{1}{2}}(\partial\Omega)$ of $u \cdot n$.

Proposition 2.1. *Let $m \in \mathbb{N}_+$ be an integer. Let $u \in H^m$ be a vector-valued function. Then, there exists a constant $C > 0$ independent of u , such that*

$$\|u\|_{H^m} \leq C(\|\nabla \times u\|_{H^{m-1}} + \|\operatorname{div} u\|_{H^{m-1}} + \|u\|_{H^{m-1}} + |u \cdot n|_{H^{m-\frac{1}{2}}(\partial\Omega)}), \quad (2.1)$$

and

$$\|u\|_{H^m} \leq C(\|\nabla \times u\|_{H^{m-1}} + \|\operatorname{div} u\|_{H^{m-1}} + \|u\|_{H^{m-1}} + |n \times u|_{H^{m-\frac{1}{2}}(\partial\Omega)}). \quad (2.2)$$

In this paper, one repeatedly use the Gagliardo-Nirenberg-Morser type inequality, whose proof can be find in [26]. First, define the space

$$W^m(\Omega \times [0, T]) = \{f(x, t) \in L^2(\Omega \times [0, T]) | \mathcal{Z}^\alpha f \in L^2(\Omega \times [0, T]), |\alpha| \leq m\}. \quad (2.3)$$

Then, the Gagliardo-Nirenberg-Morser type inequality can be stated as follows:

Proposition 2.2. *For $u, v \in L^\infty(\Omega \times [0, T]) \cap \mathcal{W}^m(\Omega \times [0, T])$ with $m \in \mathbb{N}_+$ an integer, it holds that*

$$\int_0^t \|(\mathcal{Z}^\beta u \mathcal{Z}^\gamma v)(\tau)\|_{L^2}^2 d\tau \lesssim \|u\|_{L_{t,x}^\infty}^2 \int_0^t \|v(\tau)\|_{\mathcal{H}^m}^2 d\tau + \|v\|_{L_{t,x}^\infty}^2 \int_0^t \|u(\tau)\|_{\mathcal{H}^m}^2 d\tau, \quad |\beta| + |\gamma| = m. \quad (2.4)$$

We also need the following anisotropic Sobolev embedding and trace theorems, refer to [12].

Proposition 2.3. *Let $m_1 \geq 0, m_2 \geq 0$ be integers and $f \in H_{co}^{m_1}(\Omega) \cap H_{co}^{m_2}(\Omega)$ and $\nabla f \in H_{co}^{m_2}(\Omega)$.*

(1) *The following anisotropic Sobolev embedding holds:*

$$\|f\|_{L^\infty}^2 \leq C(\|\nabla f\|_{H_{co}^{m_2}} + \|f\|_{H_{co}^{m_2}}) \cdot \|f\|_{H_{co}^{m_1}}, \quad (2.5)$$

provided $m_1 + m_2 \geq 3$.

(2) *The following trace estimate holds:*

$$\|f\|_{H^s(\partial\Omega)}^2 \leq C(\|\nabla f\|_{H_{co}^{m_2}} + \|f\|_{H_{co}^{m_2}}) \cdot \|f\|_{H_{co}^{m_1}}, \quad (2.6)$$

provided $m_1 + m_2 \geq 2s \geq 0$.

3. A PRIORI ESTIMATES

The aim of this section is to prove the following a priori estimates, which are crucial to prove Theorem (1.1). For notational convenience, we drop the superscript ε throughout this section.

Theorem 3.1 (a priori estimates). *Let m be an integer satisfying $m \geq 6$, Ω be a C^{m+2} domain, and $A \in C^{m+1}(\partial\Omega)$. For sufficiently smooth solutions defined on $[0, T]$ of (1.1) and (1.2), then it holds that*

$$|\rho(x, 0)| \exp\left(-\int_0^t \|\operatorname{div} u(\tau)\|_{L^\infty} d\tau\right) \leq \rho(x, t) \leq |\rho(x, 0)| \exp\left(\int_0^t \|\operatorname{div} u(\tau)\|_{L^\infty} d\tau\right), \quad (3.1)$$

for $(x, t) \in \Omega \times [0, T]$. In addition, if

$$0 < c_0 \leq \rho(x, t) \leq \frac{1}{c_0} < \infty, \quad (x, t) \in \Omega \times [0, T], \quad (3.2)$$

where c_0 is any given small positive constant, then the following a priori estimate holds

$$\begin{aligned} N_m(t) &+ \int_0^t (\|\nabla \partial_t^{m-1} p(\tau)\|_{L^2}^2 + \|\Delta p(\tau)\|_{\mathcal{H}^2}^2) d\tau + \varepsilon \int_0^t \|\nabla u(\tau)\|_{\mathcal{H}^m}^2 d\tau \\ &+ \varepsilon \sum_{k=0}^{m-2} \int_0^t \|\nabla^2 \partial_t^k u(\tau)\|_{m-1-k}^2 d\tau + \varepsilon^2 \int_0^t \|\nabla^2 \partial_t^{m-1} u(\tau)\|_{L^2}^2 d\tau \\ &+ \int_0^t \|\Delta d(\tau)\|_{\mathcal{H}^m}^2 d\tau + \int_0^t \|\nabla \Delta d(\tau)\|_{\mathcal{H}^{m-1}}^2 d\tau \\ &\leq \tilde{C}_2 C_{m+2} \left\{ P(N_m(0)) + P(N_m(t)) \int_0^t P(N_m(\tau)) d\tau \right\}, \quad \forall t \in [0, T], \end{aligned} \quad (3.3)$$

where \tilde{C}_2 depends only on $\frac{1}{c_0}$, $P(\cdot)$ is a polynomial, and

$$\begin{aligned} N_m(t) &\triangleq \sup_{0 \leq \tau \leq t} \left\{ 1 + \|(p, u)(\tau)\|_{\mathcal{H}^m}^2 + \|d(\tau)\|_{L^2}^2 + \|\nabla d(\tau)\|_{\mathcal{H}^m}^2 + \|\nabla u(\tau)\|_{\mathcal{H}^{m-1}}^2 \right. \\ &\quad + \|\Delta d(\tau)\|_{\mathcal{H}^{m-1}}^2 + \sum_{k=0}^{m-2} \|\partial_t^k \nabla p(\tau)\|_{m-1-k}^2 + \varepsilon \|\nabla \partial_t^{m-1} p(\tau)\|_{L^2}^2 \\ &\quad \left. + \|\Delta p(\tau)\|_{\mathcal{H}^1}^2 + \varepsilon \|\Delta p(\tau)\|_{\mathcal{H}^2}^2 + \|\nabla u(\tau)\|_{\mathcal{H}^{1,\infty}}^2 \right\}. \end{aligned} \quad (3.4)$$

Throughout this section, we shall work on the interval of time $[0, T]$ such that $c_0 \leq \rho(x, t) \leq \frac{1}{c_0}$. Furthermore, we point out that the generic constant C may depend on $\frac{1}{c_0}$ in this section. Since the proof of Theorem 3.1 is quite lengthy and involved, we divide the proof into the following several subsections.

3.1. Conormal Energy Estimates for ρ, u and ∇d . For any smooth function f , notice that

$$\Delta f = \nabla \operatorname{div} f - \nabla \times (\nabla \times f),$$

and then (1.1)₁ can be written as

$$\rho u_t + \rho u \cdot \nabla u + \nabla p = -\mu \varepsilon \nabla \times (\nabla \times u) + (2\mu + \lambda) \varepsilon \nabla \operatorname{div} u - \nabla d \cdot \Delta d. \quad (3.5)$$

In this subsection, we first give the basic a priori L^2 estimate which holds for (1.1) and (1.3).

Lemma 3.2. *For a smooth solution to (1.1) and (1.3), it holds that for $\varepsilon \in (0, 1]$*

$$\begin{aligned} & \int \left(\frac{1}{2} \rho |u|^2 + \frac{\gamma}{\gamma-1} \rho^\gamma + \frac{1}{2} |\nabla d|^2 \right) dx + c_1 \varepsilon \int_0^t \|\nabla u\|_{L^2}^2 d\tau + \int_0^t \|\Delta d\|_{L^2}^2 d\tau \\ & \leq \int \left(\frac{1}{2} \rho_0 |u_0|^2 + \frac{\gamma}{\gamma-1} \rho_0^\gamma + \frac{1}{2} |\nabla d_0|^2 \right) dx + \|\nabla d\|_{L^\infty}^2 \int_0^t \|\nabla d\|_{L^2}^2 d\tau + C_2 \int_0^t \|u\|_{L^2}^2 d\tau. \end{aligned} \quad (3.6)$$

Proof. Multiplying (1.1)₂ by d , one arrives at

$$\frac{d}{dt} \frac{1}{2} \int (|d|^2 - 1) dx + \int u \cdot \nabla (|d|^2 - 1) dx = \int \Delta d \cdot d \, dx + \int |\nabla d|^2 |d|^2 dx,$$

which, integrating by part and applying the boundary condition (1.3), yields that

$$\frac{d}{dt} \int (|d|^2 - 1) dx + 2 \int (|d|^2 - 1) (|\nabla d|^2 - \operatorname{div} u) dx = 0. \quad (3.7)$$

In view of the Grönwall inequality, one deduces from the identity (3.7) that

$$|d| = 1 \quad \text{in } \overline{\Omega}. \quad (3.8)$$

Multiplying (3.5) by u , integrating by parts and applying the boundary condition (1.3), we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho |u|^2 dx + \int \nabla p \cdot u \, dx + \mu \varepsilon \int \nabla \times (\nabla \times u) \cdot u \, dx \\ & = (2\mu + \lambda) \varepsilon \int \nabla \operatorname{div} u \cdot u \, dx - \int (u \cdot \nabla) d \cdot \Delta d \, dx. \end{aligned} \quad (3.9)$$

By virtue of the equation (1.1)₁, one deduces that

$$\int \nabla p \cdot u \, dx = \frac{\gamma}{\gamma-1} \int \nabla (\rho^{\gamma-1}) \cdot \rho u \, dx = \frac{\gamma}{\gamma-1} \int \rho^{\gamma-1} \rho_t \, dx = \frac{d}{dt} \frac{\gamma}{\gamma-1} \int \rho^\gamma \, dx. \quad (3.10)$$

Integrating by part and applying the boundary condition (1.3), we get

$$\begin{aligned} \int \nabla \times (\nabla \times u) u \, dx &= \int_{\partial\Omega} n \times (\nabla \times u) \cdot u \, d\sigma + \int |\nabla \times u|^2 \, dx \\ &= \int_{\partial\Omega} [Bu]_\tau \cdot u_\tau \, d\sigma + \int |\nabla \times u|^2 \, dx, \end{aligned}$$

and

$$\int \nabla \operatorname{div} u \cdot u \, dx = \int_{\partial\Omega} (\operatorname{div} u) u \cdot n \, d\sigma - \int |\operatorname{div} u|^2 \, dx = - \int |\operatorname{div} u|^2 \, dx. \quad (3.11)$$

which, together with (3.9), gives directly

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{1}{2} \rho |u|^2 + \frac{\gamma}{\gamma-1} \rho^\gamma \right) dx + \mu \varepsilon \int |\nabla \times u|^2 \, dx + (2\mu + \lambda) \varepsilon \int |\operatorname{div} u|^2 \, dx \\ & = -\varepsilon \int_{\partial\Omega} [Bu]_\tau \cdot u_\tau \, d\sigma - \int (u \cdot \nabla) d \cdot \Delta d \, dx. \end{aligned} \quad (3.12)$$

Multiplying (1.1)₃ by Δd , one arrives at

$$\int (d_t + u \cdot \nabla d) \cdot \Delta d \, dx = \int |\Delta d|^2 \, dx + \int |\nabla d|^2 d \cdot \Delta d \, dx. \quad (3.13)$$

Integration by part and application of boundary condition (1.3) yield directly

$$\int d_t \cdot \Delta d \, dx = \int_{\partial\Omega} d_t \cdot \frac{\partial d}{\partial n} d\sigma - \frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 dx = -\frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 dx. \quad (3.14)$$

By virtue of the basic fact $|d| = 1$, we find $\Delta d \cdot d = -|\nabla d|^2$. Then, the combination of (3.13) and (3.14) gives

$$\frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 dx + \int |\Delta d|^2 dx = \int (u \cdot \nabla) d \cdot \Delta d \, dx + \int |\nabla d|^4 dx,$$

which, together with (3.12), yields directly

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{1}{2} \rho |u|^2 + \frac{\gamma}{\gamma-1} \rho^\gamma + \frac{1}{2} |\nabla d|^2 \right) dx + \int |\Delta d|^2 dx \\ & + \mu \varepsilon \int |\nabla \times u|^2 dx + (2\mu + \lambda) \varepsilon \int |\operatorname{div} u|^2 dx \\ & = -\varepsilon \int_{\partial\Omega} [Bu]_\tau \cdot u_\tau \, d\sigma + \int |\nabla d|^4 dx. \end{aligned} \quad (3.15)$$

The trace theorem in Proposition 2.3 implies

$$|u|_{L^2(\partial\Omega)}^2 \leq \delta \|\nabla u\|_{L^2}^2 + C_\delta \|u\|_{L^2}^2. \quad (3.16)$$

The application of Proposition 2.1 gives immediately

$$\begin{aligned} & \mu \|\nabla \times u\|_{L^2}^2 + (2\mu + \lambda) \|\operatorname{div} u\|_{L^2}^2 \\ & \geq \min\{\mu, 2\mu + \lambda\} (\|\nabla \times u\|_{L^2}^2 + \|\operatorname{div} u\|_{L^2}^2) \\ & \geq 2c_1 \|\nabla u\|_{L^2}^2 - C \|u\|_{L^2}^2. \end{aligned} \quad (3.17)$$

Substituting (3.16) and (3.17) into (3.15) and choosing δ small enough, one arrives at

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{1}{2} \rho |u|^2 + \frac{\gamma}{\gamma-1} \rho^\gamma + \frac{1}{2} |\nabla d|^2 \right) dx + c_1 \varepsilon \int |\nabla u|^2 dx + \int |\Delta d|^2 dx \\ & \leq \int |\nabla d|^4 dx + C_2 \int |u|^2 dx, \end{aligned}$$

which, integrating over $[0, t]$, yields

$$\begin{aligned} & \int \left(\frac{1}{2} \rho |u|^2 + \frac{\gamma}{\gamma-1} \rho^\gamma + \frac{1}{2} |\nabla d|^2 \right) dx + c_1 \varepsilon \int_0^t \int |\nabla u|^2 dx d\tau + \int_0^t \int |\Delta d|^2 dx d\tau \\ & \leq \int \left(\frac{1}{2} \rho_0 |u_0|^2 + \frac{\gamma}{\gamma-1} \rho_0^\gamma + \frac{1}{2} |\nabla d_0|^2 \right) dx + \|\nabla d\|_{L^\infty}^2 \int_0^t \int |\nabla d|^2 dx d\tau + C_2 \int_0^t \int |u|^2 dx d\tau, \end{aligned}$$

Therefore, we complete the proof of Lemma 3.2. \square

However, the above basic energy estimation is insufficient to get the vanishing viscosity limit. Some conormal derivative estimates are needed. Let

$$Q(t) \triangleq \sup_{0 \leq \tau \leq t} \{ \|(\nabla p, \nabla u)\|_{\mathcal{H}^{1,\infty}}^2 + \|(p, u, p_t, u_t)\|_{L^\infty}^2 + \|d_t\|_{W^{1,\infty}}^2 + \|\nabla d\|_{W^{1,\infty}}^2 + \|\nabla \Delta d\|_{L^\infty}^2 \}. \quad (3.18)$$

and

$$\begin{aligned} \Lambda_m(t) & \triangleq \|(p, u, \nabla d)(t)\|_{\mathcal{H}^m}^2 + \|(\nabla u, \Delta d)(t)\|_{\mathcal{H}^{m-1}}^2 + \|\nabla u(\tau)\|_{\mathcal{H}^{1,\infty}}^2 \\ & + \sum_{k=0}^{m-2} \|\nabla \partial_t^k p(t)\|_{m-1-k}^2 + \varepsilon \|\nabla \partial_t^{m-1} p(t)\|_{L^2}^2. \end{aligned} \quad (3.19)$$

Lemma 3.3. For $m \in \mathbb{N}^+$ and a smooth solution to (1.1) and (1.3), it holds that for $\varepsilon \in (0, 1]$,

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \|(u, p, \nabla d)(\tau)\|_{\mathcal{H}^m}^2 + C\varepsilon \int_0^t \|\nabla u(\tau)\|_{\mathcal{H}^m}^2 d\tau + \int_0^t \|\Delta d(\tau)\|_{\mathcal{H}^m}^2 d\tau \\ & \leq C_{m+2} \left\{ \|(u_0, p_0, \nabla d_0)\|_{\mathcal{H}^m}^2 + \delta \int_0^t \|\nabla \Delta d(\tau)\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta \varepsilon^2 \int_0^t \|\nabla^2 u(\tau)\|_{\mathcal{H}^{m-1}}^2 d\tau \right. \\ & \quad \left. + \delta \int_0^t \|\nabla \partial_t^{m-1} p(\tau)\|_{L^2}^2 d\tau + C_\delta [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau \right\}, \end{aligned} \quad (3.20)$$

where δ is a small constant which will be chosen late, C_δ is a polynomial function of $\frac{1}{\delta}$, and the generic positive constant $C > 0$ depends on μ and λ .

Proof. The case for $m = 0$ is already proved in Lemma 3.2. Assume that (3.20) is proved for $k = m - 1$. We shall prove that it holds for $k = m \geq 1$. Applying the operator $\mathcal{Z}^\alpha (|\alpha_0| + |\alpha_1| = m)$ to the equation (3.5), we find

$$\begin{aligned} & \rho \mathcal{Z}^\alpha u_t + \rho u \cdot \nabla \mathcal{Z}^\alpha u + \mathcal{Z}^\alpha \nabla p \\ & = -\mu \varepsilon \mathcal{Z}^\alpha \nabla \times (\nabla \times u) + (2\mu + \lambda) \varepsilon \mathcal{Z}^\alpha \nabla \operatorname{div} u - \mathcal{Z}^\alpha (\nabla d \cdot \Delta d) + C_1^\alpha + C_2^\alpha, \end{aligned} \quad (3.21)$$

where

$$C_1^\alpha = -[\mathcal{Z}^\alpha, \rho] u_t, \quad C_2^\alpha = -[\mathcal{Z}^\alpha, \rho u \cdot \nabla] u.$$

Multiplying (3.21) by $\mathcal{Z}^\alpha u$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho |\mathcal{Z}^\alpha u|^2 dx + \int \mathcal{Z}^\alpha \nabla p \cdot \mathcal{Z}^\alpha u \, dx \\ & = -\mu \varepsilon \int \mathcal{Z}^\alpha \nabla \times (\nabla \times u) \cdot \mathcal{Z}^\alpha u \, dx - (2\mu + \lambda) \varepsilon \int \mathcal{Z}^\alpha \nabla \operatorname{div} u \cdot \mathcal{Z}^\alpha u \, dx \\ & \quad - \int \mathcal{Z}^\alpha (\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha u \, dx + \int C_1^\alpha \cdot \mathcal{Z}^\alpha u \, dx + \int C_2^\alpha \cdot \mathcal{Z}^\alpha u \, dx. \end{aligned} \quad (3.22)$$

Using the same argument as Lemma 3.4 of [12], one can obtain the following estimates

$$\begin{aligned} & -\varepsilon \int \mathcal{Z}^\alpha \nabla \times (\nabla \times u) \cdot \mathcal{Z}^\alpha u \, dx \\ & \leq -\frac{3\varepsilon}{4} \|\nabla \times \mathcal{Z}^\alpha u\|_{L^2}^2 + \delta \varepsilon^2 \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 + C_\delta C_{m+2} (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|u\|_{\mathcal{H}^m}^2) \end{aligned} \quad (3.23)$$

and

$$\begin{aligned} & \varepsilon \int \mathcal{Z}^\alpha \nabla \operatorname{div} u \cdot \mathcal{Z}^\alpha u \, dx \\ & \leq -\frac{3\varepsilon}{4} \|\operatorname{div} \mathcal{Z}^\alpha u\|_{L^2}^2 + \delta \varepsilon^2 \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 + C_\delta C_{m+2} (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|u\|_{\mathcal{H}^m}^2). \end{aligned} \quad (3.24)$$

On the other hand, it follows from Proposition 2.1 that

$$\begin{aligned} 2c_1 \|\nabla \mathcal{Z}^\alpha u\|_{L^2}^2 & \leq (\mu \|\nabla \times \mathcal{Z}^\alpha u\|_{L^2}^2 + (2\mu + \lambda) \|\operatorname{div} \mathcal{Z}^\alpha u\|_{L^2}^2 + \|\mathcal{Z}^\alpha u\|_{L^2}^2 + |\mathcal{Z}^\alpha u \cdot n|_{H^{\frac{1}{2}}(\partial\Omega)}) \\ & \leq (\mu \|\nabla \times \mathcal{Z}^\alpha u\|_{L^2}^2 + (2\mu + \lambda) \|\operatorname{div} \mathcal{Z}^\alpha u\|_{L^2}^2) + C_{m+2} (\|u\|_{\mathcal{H}^m}^2 + \|\nabla u\|_{\mathcal{H}^{m-1}}^2), \end{aligned} \quad (3.25)$$

where we have using the fact

$$|\mathcal{Z}^\alpha u \cdot n|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C_{m+2} (\|u\|_{\mathcal{H}^m}^2 + \|\nabla u\|_{\mathcal{H}^{m-1}}^2).$$

Substituting (3.23)-(3.25) into (3.22) and integrating the resulting inequality over $[0, t]$, we find

$$\begin{aligned}
& \frac{1}{2} \int \rho |\mathcal{Z}^\alpha u(t)|^2 dx + \frac{3c_1 \varepsilon}{2} \int_0^t \int |\nabla \mathcal{Z}^\alpha u|^2 dx d\tau + \int_0^t \int \mathcal{Z}^\alpha \nabla p \cdot \mathcal{Z}^\alpha u \, dx d\tau \\
& \leq \frac{1}{2} \int \rho_0 |\mathcal{Z}^\alpha u_0|^2 dx + C\delta_1 \varepsilon \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 d\tau + C\delta \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau \\
& \quad + C_\delta C_{m+2} \int_0^t (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|u\|_{\mathcal{H}^m}^2) d\tau + \int_0^t \int \mathcal{C}_1^\alpha \cdot \mathcal{Z}^\alpha u \, dx d\tau \\
& \quad + \int_0^t \int \mathcal{C}_2^\alpha \cdot \mathcal{Z}^\alpha u \, dx d\tau - \int_0^t \int \mathcal{Z}^\alpha (\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha u \, dx d\tau.
\end{aligned} \tag{3.26}$$

Applying the transport equation (1.1)₁, we follow the same argument as Lemma 3.4 of [12] to obtain

$$\begin{aligned}
- \int \mathcal{Z}^\alpha \nabla p \cdot \mathcal{Z}^\alpha u \, dx & \leq - \int \frac{1}{2\gamma p} |\mathcal{Z}^\alpha p|^2 dx + \int \frac{1}{2\gamma p_0} |\mathcal{Z}^\alpha p_0|^2 dx + C\delta \int_0^t \|\nabla p\|_{\mathcal{H}^{m-1}}^2 dx \\
& \quad + C_\delta [1 + P(Q(t))] \int (\|(p, u)\|_{\mathcal{H}^m}^2 + \|\nabla u\|_{\mathcal{H}^{m-1}}^2) d\tau.
\end{aligned} \tag{3.27}$$

In view of the Proposition 2.2, we obtain

$$\int_0^t \|\mathcal{Z}^\alpha (\nabla d \cdot \Delta d)\|_{L^2}^2 d\tau \leq C \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + C \|\Delta d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau$$

which, by using the Cauchy inequality, yields directly

$$\begin{aligned}
& \left| - \int_0^t \int \mathcal{Z}^\alpha (\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha u \, dx d\tau \right| \\
& \leq \delta_1 \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1} (\|\nabla d\|_{L_{x,t}^\infty}^2 + \|\Delta d\|_{L_{x,t}^\infty}^2) \int_0^t (\|u\|_{\mathcal{H}^m}^2 + \|\nabla d\|_{\mathcal{H}^m}^2) d\tau.
\end{aligned} \tag{3.28}$$

Similarly, it is easy to deduce that (or see Lemma 3.4 of [12])

$$\int_0^t \int \mathcal{C}_1^\alpha \cdot \mathcal{Z}^\alpha u \, dx d\tau \leq C[1 + P(Q(t))] \int_0^t \|(p, u)\|_{\mathcal{H}^m}^2 d\tau \tag{3.29}$$

and

$$\int_0^t \int \mathcal{C}_2^\alpha \cdot \mathcal{Z}^\alpha u \, dx d\tau \leq C[1 + P(Q(t))] \int_0^t (\|(p, u)\|_{\mathcal{H}^m}^2 + \|\nabla u\|_{\mathcal{H}^{m-1}}^2) d\tau. \tag{3.30}$$

Substituting (3.27)-(3.30) into (3.26), one attains

$$\begin{aligned}
& \frac{1}{2} \int \rho |\mathcal{Z}^\alpha u|^2 dx + \int \frac{1}{2\gamma p} |\mathcal{Z}^\alpha p|^2 dx + \frac{3c_1 \varepsilon}{2} \int_0^t \int |\nabla \mathcal{Z}^\alpha u|^2 dx d\tau \\
& \leq \frac{1}{2} \int \rho_0 |\mathcal{Z}^\alpha u_0|^2 dx + \int \frac{1}{2\gamma p_0} |\mathcal{Z}^\alpha p_0|^2 dx + C\delta_1 \varepsilon \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 d\tau + C\delta \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau \\
& \quad + C_\delta C_{m+2} [1 + P(Q(t))] \int_0^t (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|(p, u, \nabla d)\|_{\mathcal{H}^m}^2) d\tau.
\end{aligned} \tag{3.31}$$

Applying the operator $\mathcal{Z}^\alpha \nabla (|\alpha_0| + |\alpha_1| = m)$ to the equation (1.1)₃, we find

$$\mathcal{Z}^\alpha \nabla d_t - \mathcal{Z}^\alpha \nabla \Delta d = -\mathcal{Z}^\alpha \nabla (u \cdot \nabla d) + \mathcal{Z}^\alpha \nabla (|\nabla d|^2 d). \tag{3.32}$$

Multiplying (3.32) by $\mathcal{Z}^\alpha \nabla d$, it is easy to deduce that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int |\mathcal{Z}^\alpha \nabla d|^2 dx - \int \mathcal{Z}^\alpha \nabla \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx \\
& = - \int \mathcal{Z}^\alpha \nabla (u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx + \int \mathcal{Z}^\alpha \nabla (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx.
\end{aligned} \tag{3.33}$$

Integrating by part, it is easy to check that

$$\begin{aligned}
& - \int \mathcal{Z}^\alpha \nabla \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx \\
& = - \int \nabla \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx - \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx \\
& = - \int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma + \int \mathcal{Z}^\alpha \Delta d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx - \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx \\
& = - \int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma + \int |\operatorname{div}(\mathcal{Z}^\alpha \nabla d)|^2 \, dx + \int [\mathcal{Z}^\alpha, \operatorname{div}] \nabla d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx \\
& \quad - \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx.
\end{aligned}$$

This, together with (3.33), reads

$$\begin{aligned}
& \frac{1}{2} \int |\mathcal{Z}^\alpha \nabla d(t)|^2 \, dx + \int_0^t \int |\operatorname{div}(\mathcal{Z}^\alpha \nabla d)|^2 \, dx \\
& = \frac{1}{2} \int |\mathcal{Z}^\alpha \nabla d_0|^2 \, dx - \int_0^t \int \mathcal{Z}^\alpha \nabla(u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\
& \quad + \int_0^t \int \mathcal{Z}^\alpha \nabla(|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau - \int_0^t \int [\mathcal{Z}^\alpha, \operatorname{div}] \nabla d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \\
& \quad + \int_0^t \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau + \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau \\
& \triangleq I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{3.34}$$

Deal with the term I_2 . Integrating by part, one arrives at

$$\begin{aligned}
I_2 & = - \int_0^t \int \nabla \mathcal{Z}^\alpha(u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau - \int_0^t \int [\mathcal{Z}^\alpha, \nabla](u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\
& = - \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha(u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau - \int_0^t \int \mathcal{Z}^\alpha(u \cdot \nabla d) \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \\
& \quad - \int_0^t \int [\mathcal{Z}^\alpha, \nabla](u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau.
\end{aligned} \tag{3.35}$$

To estimate the boundary term on the right hand side of (3.35). If $|\alpha_0| = m$, we apply the boundary condition (1.3) to deduce that

$$- \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha(u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau = 0.$$

If $|\alpha_{13}| \neq 0$, the proposition of (1.7) implies $\mathcal{Z}^\alpha \nabla d = 0$ on the boundary. Then, one arrives at

$$- \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha(u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau = 0.$$

Hence, we deal with the case of $|\alpha_{13}| = 0$ and $|\alpha_0| \leq m - 1$. For $|\beta| = m - 1 - \alpha_0$ ($|\alpha_0| \leq m - 1$), we integrating by part along the boundary to deduce that

$$- \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha(u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau \leq \int_0^t |\partial_t^{\alpha_0} Z_y^\beta(u \cdot \nabla d)|_{L^2(\partial\Omega)} |\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)} \, d\tau. \tag{3.36}$$

Applying the trace theorem in Proposition 2.3 and the Proposition 2.2, one arrives at

$$\begin{aligned}
& \int_0^t |\partial_t^{\alpha_0} Z_y^\beta(u \cdot \nabla d)|_{L^2(\partial\Omega)}^2 d\tau \\
& \leq C \int_0^t (\|\nabla \partial_t^{\alpha_0}(u \cdot \nabla d)\|_{m-1-\alpha_0}^2 + \|\partial_t^{\alpha_0}(u \cdot \nabla d)\|_{m-1-\alpha_0}^2) d\tau \\
& \leq C \int_0^t (\|\nabla(u \cdot \nabla d)\|_{\mathcal{H}^{m-1}}^2 + \|u \cdot \nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau \\
& \leq CQ(t) \int_0^t (\|(u, \nabla d)\|_{\mathcal{H}^{m-1}}^2 + \|\nabla(u, \nabla d)\|_{\mathcal{H}^{m-1}}^2) d\tau.
\end{aligned} \tag{3.37}$$

With the help of boundary condition (1.3) and trace theorem in Proposition 2.3, we find

$$\int_0^t |\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)}^2 d\tau \leq C_{m+2} \int_0^t (\|\nabla^2 d\|_{\mathcal{H}^m}^2 + \|\nabla d\|_{\mathcal{H}^m}^2) d\tau. \tag{3.38}$$

The combination of (3.36)-(3.38) and Cauchy inequality, it is easy to deduce that

$$\begin{aligned}
& - \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha(u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau \\
& \leq \delta_1 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1} C_{m+2} (1 + Q(t)) \int_0^t (\|(u, \nabla d)\|_{\mathcal{H}^m}^2 + \|\nabla(u, \nabla d)\|_{\mathcal{H}^{m-1}}^2) d\tau.
\end{aligned} \tag{3.39}$$

Applying the Young inequality and the Proposition 2.2, one arrives at

$$\begin{aligned}
& - \int_0^t \int \mathcal{Z}^\alpha(u \cdot \nabla d) \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \\
& \leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\|_{L^2}^2 d\tau + C_{\delta_1} \|u\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1} \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|u\|_{\mathcal{H}^m}^2 d\tau \\
& \leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\|_{L^2}^2 d\tau + C_{\delta_1} C_1 Q(t) \int_0^t (\|u\|_{\mathcal{H}^m}^2 + \|\nabla d\|_{\mathcal{H}^m}^2) d\tau
\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}
& - \int_0^t \int [\mathcal{Z}^\alpha, \nabla](u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\
& \leq \sum_{|\beta| \leq m-1} \int_0^t \|\mathcal{Z}^\beta(\nabla u \cdot \nabla d + u \cdot \nabla^2 d)\|_{L^2} \|\mathcal{Z}^\alpha \nabla d\|_{L^2} d\tau \\
& \leq C \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau + C(\|\nabla u\|_{L_{x,t}^\infty}^2 + \|\nabla d\|_{L_{x,t}^\infty}^2) \int_0^t (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau \\
& \quad + C(\|u\|_{L_{x,t}^\infty}^2 + \|\nabla^2 d\|_{L_{x,t}^\infty}^2) \int_0^t (\|u\|_{\mathcal{H}^{m-1}}^2 + \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2) d\tau \\
& \leq C(1 + Q(t)) \int_0^t (\|u\|_{\mathcal{H}^{m-1}}^2 + \|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^m}^2 + \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2) d\tau.
\end{aligned} \tag{3.41}$$

Substituting (3.39)-(3.41) into (3.35), we obtain

$$|I_2| \leq \delta_1 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1} C_1 (1 + Q(t)) \int_0^t (\|(u, \nabla d)\|_{\mathcal{H}^m}^2 + \|\nabla(u, \nabla d)\|_{\mathcal{H}^{m-1}}^2) d\tau. \tag{3.42}$$

Deal with the term I_3 . Indeed, by integrating by part, one arrives at

$$\begin{aligned}
I_3 &= \int_0^t \int \nabla \mathcal{Z}^\alpha (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau + \int_0^t \int [\mathcal{Z}^\alpha, \nabla] (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\
&= - \int_0^t \int \mathcal{Z}^\alpha (|\nabla d|^2 d) \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau + \int_0^t \int [\mathcal{Z}^\alpha, \nabla] (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\
&\quad + \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau.
\end{aligned} \tag{3.43}$$

It is easy to deduce that

$$\begin{aligned}
&- \int_0^t \int \mathcal{Z}^\alpha (|\nabla d|^2 d) \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \\
&= - \sum_{|\beta| \geq 1} \int_0^t \int \mathcal{Z}^\gamma (|\nabla d|^2) \mathcal{Z}^\beta d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \\
&\quad - \int_0^t \int \mathcal{Z}^\alpha (|\nabla d|^2) d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau.
\end{aligned} \tag{3.44}$$

By virtue of the Proposition 2.2, we obtain

$$\begin{aligned}
&\sum_{|\beta| \geq 1} \int_0^t \|\mathcal{Z}^\gamma (|\nabla d|^2) \mathcal{Z}^\beta d\|_{L^2}^2 d\tau \\
&\leq \|\mathcal{Z} d\|_{L_{x,t}^\infty}^2 \int_0^t \| |\nabla d|^2 \|_{\mathcal{H}^{m-1}}^2 d\tau + \| |\nabla d|^2 \|_{L_{x,t}^\infty}^2 \int_0^t \|\mathcal{Z} d\|_{\mathcal{H}^{m-1}}^2 d\tau \\
&\leq \|\mathcal{Z} d\|_{L^\infty}^2 \|\nabla d\|_{L^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + \|\nabla d\|_{L^\infty}^4 \int_0^t \|\mathcal{Z} d\|_{\mathcal{H}^{m-1}}^2 d\tau \\
&\leq \|\nabla d\|_{L^\infty}^4 \int_0^t \|\partial_t d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_1(1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau.
\end{aligned} \tag{3.45}$$

By virtue of equation (1.1)₂, we find

$$\begin{aligned}
\int_0^t \|\partial_t d\|_{\mathcal{H}^{m-1}}^2 d\tau &\leq \|u\|_{L^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + \|\nabla d\|_{L^\infty}^2 \int_0^t \|u\|_{\mathcal{H}^{m-1}}^2 d\tau \\
&\quad + \int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \| |\nabla d|^2 d \|_{\mathcal{H}^{m-1}}^2 d\tau.
\end{aligned} \tag{3.46}$$

By virtue of the Proposition 2.2, we obtain

$$\begin{aligned}
\int_0^t \| |\nabla d|^2 d \|_{\mathcal{H}^{m-1}}^2 d\tau &\leq \sum_{|\gamma| \geq 1, |\beta| + |\gamma| \leq m-1} \int_0^t \|\mathcal{Z}^\beta (|\nabla d|^2) \mathcal{Z}^\gamma d\|_{L^2}^2 d\tau + \int_0^t \| |\nabla d|^2 \|_{\mathcal{H}^{m-1}}^2 d\tau \\
&\lesssim \|\mathcal{Z} d\|_{L^\infty}^2 \|\nabla d\|_{L^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-2}}^2 d\tau + \|\nabla d\|_{L^\infty}^4 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-2}}^2 d\tau \\
&\quad + \|\nabla d\|_{L^\infty}^4 \int_0^t \|\partial_t d\|_{\mathcal{H}^{m-2}}^2 d\tau + \|\nabla d\|_{L^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau.
\end{aligned} \tag{3.47}$$

Substituting (3.47) into (3.46), one arrives at immediately

$$\begin{aligned}
\int_0^t \|d_t\|_{\mathcal{H}^{m-1}}^2 d\tau &\lesssim \|\nabla d\|_{L^\infty}^4 \int_0^t \|\partial_t d\|_{\mathcal{H}^{m-2}}^2 d\tau + \int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\
&\quad + C_1(1 + P(Q(t))) \int_0^t (\|u\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau.
\end{aligned} \tag{3.48}$$

On the other hand, it is easy to deduce that

$$\int_0^t \|d_t\|_{L^2}^2 d\tau \lesssim \int_0^t \|\Delta d\|_{L^2}^2 d\tau + (1 + \|\nabla d\|_{L^\infty}^2) \int_0^t (\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) d\tau. \quad (3.49)$$

The combination of (3.48) and (3.49) yields directly

$$\begin{aligned} \int_0^t \|d_t\|_{\mathcal{H}^{m-1}}^2 d\tau &\leq C_1(1 + P(Q(t))) \int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\ &\quad + C_1(1 + P(Q(t))) \int_0^t (\|u\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau, \end{aligned} \quad (3.50)$$

which, together with (3.45), gives directly

$$\begin{aligned} &\left| - \sum_{|\beta| \geq 1} \int_0^t \int \mathcal{Z}^\gamma (|\nabla d|^2) \mathcal{Z}^\beta d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \right| \\ &\leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\|_{L^2} d\tau + C_{\delta_1} C_1(1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau. \end{aligned} \quad (3.51)$$

In view of the Proposition 2.2 and Cauchy inequality, we obtain

$$\begin{aligned} &\left| - \int_0^t \int \mathcal{Z}^\alpha (|\nabla d|^2) d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \right| \\ &\leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\|_{L^2} d\tau + C_{\delta_1} \|\nabla d\|_{L^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau. \end{aligned} \quad (3.52)$$

Then combination of (3.51) and (3.52) yields immediately

$$\begin{aligned} &- \int_0^t \int \mathcal{Z}^\alpha (|\nabla d|^2) d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \\ &\leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\|_{L^2} d\tau + C_1 C_{\delta_1} (1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau. \end{aligned} \quad (3.53)$$

On the other hand, we find that

$$\begin{aligned} &\int_0^t \int [\mathcal{Z}^\alpha, \nabla] (|\nabla d|^2) d \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\ &= \sum_{|\beta| \leq m-1} \int_0^t \int d \cdot \mathcal{Z}^\beta (\nabla d \cdot \nabla^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\ &\quad + \sum_{\substack{|\beta| \geq 1 \\ |\beta| + |\gamma| \leq m-1}} \int_0^t \int \mathcal{Z}^\beta d \cdot \mathcal{Z}^\gamma (\nabla d \cdot \nabla^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\ &\quad + \sum_{|\beta| + |\gamma| \leq m-1} \int_0^t \int \mathcal{Z}^\beta (|\nabla d|^2) \mathcal{Z}^\gamma \nabla d \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\ &= II_1 + II_2 + II_3. \end{aligned} \quad (3.54)$$

In view of the Proposition 2.2, we find

$$II_1 \lesssim \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau + \|\nabla^2 d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau. \quad (3.55)$$

and

$$\begin{aligned}
I_2 &\lesssim \|\nabla d\|_{W_{x,t}^{1,\infty}}^4 \int_0^t \|\mathcal{Z}d\|_{\mathcal{H}^{m-2}}^2 d\tau + \|\mathcal{Z}d\|_{L_{x,t}^\infty}^2 \|\nabla d\|_{L^\infty}^2 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^{m-2}}^2 d\tau \\
&\quad + \|\mathcal{Z}d\|_{L_{x,t}^\infty}^2 \|\nabla^2 d\|_{L^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-2}}^2 d\tau + \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau \\
&\leq C_1(1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau,
\end{aligned} \tag{3.56}$$

where we have used the estimate (3.50). Similarly, it is easy to deduce that

$$I_3 \leq C\|\nabla d\|_{L^\infty}^4 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + C \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau. \tag{3.57}$$

Substituting (3.55)-(3.57) into (3.54), one arrives at

$$\int_0^t \int [\mathcal{Z}^\alpha, \nabla](|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \leq C_1(1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau. \tag{3.58}$$

Deal with the boundary term on the right hand side of (3.43). If $|\alpha_0| = m$ or $|\alpha_{13}| \geq 1$, we obtain

$$\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha(|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau = 0. \tag{3.59}$$

On the other hand, it is easy to deduce that for $|\beta| = m - 1 - \alpha_0$

$$\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha(|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau \leq \int_0^t |\partial_t^{\alpha_0} \mathcal{Z}_y^\beta(|\nabla d|^2 d)|_{L^2(\partial\Omega)} |\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)} d\tau. \tag{3.60}$$

By virtue of the trace theorem in Proposition 2.3, we find for $|\beta| = m - 1 - \alpha_0$

$$\begin{aligned}
&|\partial_t^{\alpha_0} \mathcal{Z}_y^\beta(|\nabla d|^2 d)|_{L^2(\partial\Omega)}^2 \\
&\leq C\|\nabla \partial_t^{\alpha_0}(|\nabla d|^2 d)\|_{m-1-\alpha_0} \|\partial_t^{\alpha_0}(|\nabla d|^2 d)\|_{m-1-\alpha_0} + C\|\partial_t^{\alpha_0}(|\nabla d|^2 d)\|_{m-1-\alpha_0}^2 \\
&\leq C\|\nabla(|\nabla d|^2 d)\|_{\mathcal{H}^{m-1}} \|\nabla d\|_{\mathcal{H}^{m-1}}^2 + C\|\nabla d\|_{\mathcal{H}^{m-1}}^2,
\end{aligned} \tag{3.61}$$

and

$$|\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)}^2 \leq C_{m+2}(\|\nabla^2 d\|_{\mathcal{H}^m}^2 + \|\nabla d\|_{\mathcal{H}^m}^2). \tag{3.62}$$

On the other hand, we obtain just following the idea as (3.44) and (3.54) that

$$\int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau \leq C(1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau \tag{3.63}$$

and

$$\int_0^t \|\nabla(|\nabla d|^2 d)\|_{\mathcal{H}^{m-1}}^2 d\tau \leq C(1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau. \tag{3.64}$$

The combination of (3.60)-(3.64) gives directly

$$\begin{aligned}
&\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha(|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau \\
&\leq \delta_1 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1} C_{m+2}(1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau.
\end{aligned} \tag{3.65}$$

Substituting (3.53), (3.58) and (3.65) into (3.43), we attains

$$|I_3| \leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\|_{L^2}^2 d\tau + \delta_1 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1} C_{m+2}(1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau. \tag{3.66}$$

Deal with the term I_4 and I_5 . In view of the Cauchy inequality, it is easy to deduce that

$$|I_4| \leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\|_{L^2}^2 d\tau + C_{\delta_1} \int_0^t \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau, \tag{3.67}$$

and

$$|I_5| \leq \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau. \quad (3.68)$$

Deal with the term I_6 . If $|\alpha_0| = m$ or $|\alpha_{13}| \geq 1$, it is easy to deduce that

$$\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n d\sigma d\tau = 0. \quad (3.69)$$

For the case of $|\alpha_0| \leq m-1$ or $|\alpha_{13}| = 0$, integrating by part along the boundary, we have for $|\beta| = m-1-\alpha_0$

$$\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n d\sigma d\tau \leq C \int_0^t |\partial_t^{\alpha_0} Z_y^\beta \Delta d|_{L^2(\partial\Omega)} |\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)} d\tau. \quad (3.70)$$

By virtue of the trace theorem in Proposition 2.3, one arrives at

$$\begin{aligned} |\partial_t^{\alpha_0} Z_y^\beta \Delta d|_{L^2(\partial\Omega)} &\leq C(\|\nabla \partial_t^{\alpha_0} Z_y^\beta \Delta d\|^{\frac{1}{2}} + \|\partial_t^{\alpha_0} Z_y^\beta \Delta d\|^{\frac{1}{2}}) \|\partial_t^{\alpha_0} Z_y^\beta \Delta d\|^{\frac{1}{2}} \\ &\leq C(\|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^{\frac{1}{2}} + \|\Delta d\|_{\mathcal{H}^{m-1}}^{\frac{1}{2}}) \|\Delta d\|_{\mathcal{H}^{m-1}}^{\frac{1}{2}}. \end{aligned} \quad (3.71)$$

Similarly, in view of boundary condition (1.3) and trace theorem in Proposition 2.3, one attains

$$|\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)} \leq C_{m+2}(\|\nabla^2 d\|_{\mathcal{H}^m}^{\frac{1}{2}} + \|\nabla d\|_{\mathcal{H}^m}^{\frac{1}{2}}) \|\nabla d\|_{\mathcal{H}^m}^{\frac{1}{2}}. \quad (3.72)$$

Substituting (3.71) and (3.72) into (3.70) and applying the Cauchy inequality, we find

$$|I_6| \leq \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta_1 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta, \delta_1} C_{m+2} \int_0^t (\|\nabla d\|_{\mathcal{H}^m}^2 + \|\Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau. \quad (3.73)$$

Substituting (3.42), (3.66)-(3.68) and (3.73) into (3.34) and choosing δ_1 small enough, we find

$$\begin{aligned} &\frac{1}{2} \int |\mathcal{Z}^\alpha \nabla d(t)|^2 dx + \frac{3}{4} \int_0^t \int |\mathcal{Z}^\alpha \Delta d|^2 dx d\tau \\ &\leq \frac{1}{2} \int |\mathcal{Z}^\alpha \nabla d_0|^2 dx + \delta_2 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\ &\quad + C_{\delta, \delta_2} C_{m+2} (1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau. \end{aligned} \quad (3.74)$$

In view of the standard elliptic regularity results with Neumann boundary condition, we get that

$$\begin{aligned} \|\nabla^2 d\|_{\mathcal{H}^m}^2 &= \|\nabla^2 \partial_t^{\alpha_0} d\|_{m-\alpha_0}^2 \\ &\leq C_{m+2} (\|\nabla \partial_t^{\alpha_0} d\|_{L^2}^2 + \|\Delta \partial_t^{\alpha_0} d\|_{m-\alpha_0}^2) \\ &\leq C_{m+2} (\|\nabla d\|_{\mathcal{H}^m}^2 + \|\Delta d\|_{\mathcal{H}^m}^2). \end{aligned} \quad (3.75)$$

The combination of (3.104), (3.74) and (3.75) yields directly

$$\begin{aligned} &\sup_{0 \leq \tau \leq t} \|(u, p, \nabla d)\|_{\mathcal{H}^m}^2 + C\varepsilon \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 d\tau + \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau \\ &\leq C_{m+2} \left\{ \|(u_0, p_0, \nabla d_0)\|_{\mathcal{H}^m}^2 + \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta \varepsilon^2 \int \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau \right. \\ &\quad \left. + \delta \int_0^t \|\nabla \partial_t^{m-1} p(\tau)\|_{L^2}^2 d\tau + C_\delta [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau \right\}. \end{aligned}$$

Therefore, we complete the proof of Lemma 3.3. \square

3.2. Normal Derivatives Estimates. In order to estimate $\|\nabla u\|_{\mathcal{H}^{m-1}}$, it remains to estimate $\|\chi_j \partial_n u\|_{\mathcal{H}^{m-1}}$, where χ_j is supported compactly in one of the Ω_j and with value one in a neighborhood of the boundary. Indeed, it follows from the definition of the norm that $\|\chi \partial_{y_i} u\|_{\mathcal{H}^{m-1}} \leq C\|u\|_{\mathcal{H}^m}$, $i = 1, 2$. Then, it suffices to estimate $\|\chi \partial_n u\|_{\mathcal{H}^{m-1}}$.

Note that

$$\operatorname{div} u = \partial_n u \cdot n + (\Pi \partial_{y_1} u)_1 + (\Pi \partial_{y_2} u)_2 \quad (3.76)$$

and

$$\partial_n u = (\partial_n u \cdot n)n + \Pi(\partial_n u). \quad (3.77)$$

Then, it follows from (3.76) and (3.77) that

$$\begin{aligned} \|\chi \partial_n u\|_{\mathcal{H}^{m-1}} &\leq \|\chi \partial_n u \cdot n\|_{\mathcal{H}^{m-1}} + \|\chi \Pi(\partial_n u)\|_{\mathcal{H}^{m-1}} \\ &\leq C_m \{ \|\chi \operatorname{div} u\|_{\mathcal{H}^{m-1}} + \|\chi \Pi(\partial_n u)\|_{\mathcal{H}^{m-1}} + \|u\|_{\mathcal{H}^m} \}. \end{aligned}$$

Thus, it suffices to estimate $\|\chi \Pi(\partial_n u)\|_{\mathcal{H}^{m-1}}$ and $\|\chi \operatorname{div} u\|_{\mathcal{H}^{m-1}}$, since $\|u\|_{\mathcal{H}^m}$ has been estimated before (see Lemma 3.3). We extend the smooth symmetric matrix A to be $A(y, z) = A(y)$. Define

$$\eta = \chi(w \times n + \Pi(Bu)) = \chi(\Pi(w \times n) + \Pi(Bu)). \quad (3.78)$$

In view of the Navier-slip boundary condition (1.3), η satisfies

$$\eta|_{\partial\Omega} = 0. \quad (3.79)$$

Since $w \times n = (\nabla u - (\nabla u)^t) \cdot n$, then η can be rewritten as

$$\eta = \chi \{ \Pi(\partial_n u) - \Pi(\nabla(u \cdot n)) + \Pi((\nabla n)^t \cdot u) + \Pi(Bu) \},$$

which, yields immediately that

$$\|\chi \Pi(\partial_n u)\|_{\mathcal{H}^{m-1}} \leq C_{m+1} (\|\eta\|_{\mathcal{H}^{m-1}} + \|u\|_{\mathcal{H}^m}). \quad (3.80)$$

Hence, it remains to estimate $\|\eta\|_{\mathcal{H}^{m-1}}$.

Lemma 3.4. *For $m \geq 1$, it holds that*

$$\begin{aligned} &\sup_{0 \leq \tau \leq t} \|\eta(\tau)\|_{L^2}^2 + \varepsilon \int_0^t \|\nabla \eta(\tau)\|_{L^2}^2 d\tau \\ &\leq CC_3 \left\{ \|u_0\|_{H^1}^2 + \delta \varepsilon^2 \int_0^t \|\nabla^2 u(\tau)\|_{L^2}^2 d\tau + \delta \int_0^t \|\nabla \Delta d\|_{L^2}^2 d\tau \right\} \\ &\quad + C_3 C_\delta [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \end{aligned} \quad (3.81)$$

Proof. Notice that

$$\nabla \times ((u \cdot \nabla)u) = (u \cdot \nabla)w - (w \cdot \nabla)u + w \operatorname{div} u,$$

so w satisfies the following equations:

$$\rho w_t + \rho(u \cdot \nabla)w = \mu \varepsilon \Delta w + F_1, \quad (3.82)$$

where

$$F_1 \triangleq -\nabla \rho \times u_t - \nabla \rho \times (u \cdot \nabla)u + \rho(w \cdot \nabla)u - \rho w \operatorname{div} u - \nabla \times (\nabla d \cdot \Delta d).$$

Consequently, the system for η is

$$\begin{aligned} &\rho \eta_t + \rho u_1 \partial_{y_1} \eta + \rho u_2 \partial_{y_2} \eta + \rho u \cdot N \partial_z \eta - \mu \varepsilon \Delta \eta \\ &= \chi[F_1 \times n + \Pi(BF_2)] + \chi F_3 + F_4 - \mu \varepsilon \chi \Delta(\Pi B) \cdot u, \end{aligned} \quad (3.83)$$

where

$$\begin{aligned}
F_2 &= (\mu + \lambda)\varepsilon \nabla \operatorname{div} u - \nabla p - \nabla d \cdot \Delta d, \\
F_3 &= -2\mu\varepsilon \sum_{i=1}^2 \partial_j w \times \partial_j n - \mu\varepsilon w \times \Delta n + \sum_{j=1}^2 \rho u_i w \times \partial_i n \\
&\quad + \sum_{i=1}^2 \rho u_i \partial_i (\Pi B) u - 2\mu\varepsilon \sum_{i=1}^2 \partial_i (\Pi B) \partial_i u, \\
F_4 &= \sum_{i=1}^2 \rho u_i \partial_{y_i} \chi \cdot (w \times n + \Pi(Bu)) + \rho u \cdot N \partial_z \chi \cdot (w \times n + \Pi(Bu)) \\
&\quad - 2\mu\varepsilon \sum_{i=1}^3 \partial_i \chi \partial_i (w \times n + \Pi(Bu)) - \mu\varepsilon \Delta \chi \cdot (w \times n + \Pi(Bu)).
\end{aligned}$$

Multiplying (3.83) by η , it is easy to check that

$$\frac{1}{2} \frac{d}{dt} \int |\eta|^2 dx + \varepsilon \int |\nabla \eta|^2 dx = \int F \cdot \eta dx - \mu\varepsilon \int \chi \Delta (\Pi B) \cdot u \cdot \eta dx, \quad (3.84)$$

where $F \triangleq \chi[F_1 \times n + \Pi(BF_2)] + \chi F_3 + F_4$. It is easy to deduce that

$$\begin{aligned}
\|\chi F_1 \times n\|_{L^2} &\leq C_2 \{[1 + P(Q(t))](\|\nabla u\|_{L^2} + \|\nabla p\|_{L^2}) + \|\nabla d\|_{L^\infty} \|\nabla \Delta d\|_{L^2}\}, \\
\|\chi \Pi(BF_2)\|_{L^2} &\leq C_2(\varepsilon \|\nabla^2 u\|_{L^2} + \|\nabla d\|_{L^\infty} \|\Delta d\|_{L^2} + \|\nabla p\|_{L^2}), \\
\|\chi F_3\|_{L^2} &\leq \varepsilon \|\nabla^2 u\|_{L^2} + C_3(1 + \|u\|_{L^\infty})(\|u\|_{L^2} + \|\nabla u\|_{L^2}).
\end{aligned} \quad (3.85)$$

Notice that the term F_4 are supported away from the boundary, we can control all the derivatives by the $\|\cdot\|_{\mathcal{H}^m}$. Hence, we find

$$\|F_4\|_{L^2} \leq \varepsilon \|\nabla^2 u\|_{L^2} + C_3(1 + \|u\|_{L^\infty})\|u\|_{\mathcal{H}^1}. \quad (3.86)$$

Integrating by parts, it is easy to deduce that

$$- \mu\varepsilon \int \chi \Delta (\Pi B) \cdot u \cdot \eta dx \leq \delta \varepsilon \int |\nabla \eta|^2 dx + C_\delta C_3(\|\nabla u\|_{L^2}^2 + \|u\|_{\mathcal{H}^1}^2). \quad (3.87)$$

Substituting (3.85)-(3.87) into (3.84) and integrating the resulting inequality over $[0, t]$, we have

$$\begin{aligned}
&\frac{1}{2} \int |\eta|^2(t) dx + \varepsilon \int_0^t \int |\nabla \eta|^2 dx d\tau \\
&\leq \frac{1}{2} \int |\eta_0|^2 dx + \delta \varepsilon^2 \int_0^t \|\nabla^2 u\|_{L^2}^2 d\tau + \delta \int_0^t \|\nabla \Delta d\|_{L^2}^2 d\tau + C[1 + P(Q(t))] \int_0^t \Lambda_1(\tau) d\tau.
\end{aligned}$$

Therefore, we complete the proof of Lemma 3.4. \square

Lemma 3.5. *For $m \geq 1$, it holds that*

$$\begin{aligned}
&\sup_{0 \leq \tau \leq t} \|\eta(\tau)\|_{\mathcal{H}^{m-1}}^2 + \mu\varepsilon \int_0^t \|\nabla \eta(\tau)\|_{\mathcal{H}^{m-1}}^2 d\tau \\
&\leq CC_{m+2} \left\{ \|(u_0, \nabla u_0)\|_{\mathcal{H}^{m-1}}^2 + \delta \int_0^t \|\nabla \partial_t^{m-1} p\|_{L^2}^2 d\tau + \delta \varepsilon^2 \int_0^t \|\nabla^2 u(\tau)\|_{\mathcal{H}^{m-1}}^2 d\tau \right\} \\
&\quad + CC_{m+2} \left\{ \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau \right\}.
\end{aligned} \quad (3.88)$$

Proof. The case for $m = 1$ is already proved in Lemma 3.4. Assume that (3.88) is proved for $k = m - 2$. We shall prove that it holds for $k = m - 1 \geq 1$. For $|\alpha| = m - 1$, applying the operator \mathcal{Z}^α to the equation (3.83), we find

$$\rho \mathcal{Z}^\alpha \eta_t + \rho(u \cdot \nabla) \mathcal{Z}^\alpha \eta - \mu\varepsilon \mathcal{Z}^\alpha \Delta \eta = \mathcal{Z}^\alpha F - \mathcal{Z}^\alpha [\mu\varepsilon \chi (\Delta (\Pi B) \cdot u)] + \mathcal{C}_3^\alpha + \mathcal{C}_4^\alpha, \quad (3.89)$$

where

$$\begin{aligned} \mathcal{C}_3^\alpha &= - \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} C_{\alpha, \beta} \mathcal{Z}^\beta \rho \mathcal{Z}^\gamma \eta_t, \\ \mathcal{C}_4^\alpha &= - \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} \sum_{i=1}^2 C_{\alpha, \beta} \mathcal{Z}^\beta (\rho u_i) \mathcal{Z}^\gamma \partial_{y_i} \eta, \\ &\quad - \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} C_{\alpha, \beta} \mathcal{Z}^\beta (\rho u \cdot N) \mathcal{Z}^\gamma \partial_z \eta, \\ &\quad - \rho(u \cdot N) \sum_{|\beta| \geq m-2} C(\alpha, \beta, z) \partial_z \mathcal{Z}^\beta \eta, \end{aligned}$$

where $C(\alpha, \beta, z)$ is smooth function depending on α, β and $\varphi(z)$. Multiplying (3.89) by $\mathcal{Z}^\alpha \eta$, it is easy to deduce that

$$\begin{aligned} &\frac{1}{2} \int \rho |\mathcal{Z}^\alpha \eta(t)|^2 dx - \frac{1}{2} \int \rho_0 |\mathcal{Z}^\alpha \eta_0|^2 dx \\ &= \mu \varepsilon \int_0^t \int \mathcal{Z}^\alpha \Delta \eta \cdot \mathcal{Z}^\alpha \eta \, dx d\tau + \int_0^t \int \mathcal{Z}^\alpha F \cdot \mathcal{Z}^\alpha \eta \, dx d\tau \\ &\quad - \mu \varepsilon \int_0^t \int \mathcal{Z}^\alpha (\chi \Delta(\Pi B) \cdot u) \cdot \mathcal{Z}^\alpha \eta \, dx d\tau + \int_0^t \int (\mathcal{C}_3^\alpha + \mathcal{C}_4^\alpha) \cdot \mathcal{Z}^\alpha \eta \, dx d\tau. \end{aligned} \quad (3.90)$$

In the local basis, it holds that

$$\partial_j = \beta_j^1 \partial_{y_1} + \beta_j^2 \partial_{y_2} + \beta_j^3 \partial_z, \quad j = 1, 2, 3,$$

for harmless functions β_j^i , $i, j = 1, 2, 3$ depending on the boundary regularity and weighted function $\varphi(z)$. Therefore, the following commutation expansion holds:

$$\mathcal{Z}^\alpha \Delta \eta = \Delta \mathcal{Z}^\alpha \eta + \sum_{|\beta| \leq m-2} C_{1\beta} \partial_{zz} \mathcal{Z}^\beta \eta + \sum_{|\beta| \leq m-1} (C_{2\beta} \partial_z \mathcal{Z}^\beta \eta + C_{3\beta} Z_y \mathcal{Z}^\beta \eta).$$

Then integrating by part and applying the Cauchy inequality, we obtain

$$\begin{aligned} &\mu \varepsilon \int_0^t \int \mathcal{Z}^\alpha \Delta \eta \cdot \mathcal{Z}^\alpha \eta \, dx d\tau \\ &= \mu \varepsilon \int_0^t \int \Delta \mathcal{Z}^\alpha \eta \cdot \mathcal{Z}^\alpha \eta \, dx d\tau + \sum_{|\beta| \leq m-2} \mu \varepsilon \int_0^t \int C_{1\beta} \partial_{zz} \mathcal{Z}^\beta \eta \cdot \mathcal{Z}^\alpha \eta \, dx d\tau \\ &\quad + \sum_{|\beta| \leq m-1} \mu \varepsilon \int_0^t \int (C_{2\beta} \partial_z \mathcal{Z}^\beta \eta + C_{3\beta} Z_y \mathcal{Z}^\beta \eta) \cdot \mathcal{Z}^\alpha \eta \, dx d\tau \\ &\leq -\frac{3}{4} \mu \varepsilon \int_0^t \|\nabla \mathcal{Z}^\alpha \eta\|_{L^2}^2 d\tau + C \mu \varepsilon \int_0^t \|\nabla \eta\|_{\mathcal{H}^{m-2}}^2 d\tau + C_{m+2} \mu \varepsilon \int_0^t \|\eta\|_{\mathcal{H}^{m-1}}^2 d\tau. \end{aligned} \quad (3.91)$$

Note that there is no boundary term in the integrating by parts since $\mathcal{Z}^\alpha \eta$ vanishes on the boundary. Substituting (3.91) into (3.90), we find

$$\begin{aligned} &\frac{1}{2} \int |\mathcal{Z}^\alpha \eta(t)|^2 dx + \frac{3}{4} \mu \varepsilon \int_0^t \|\nabla \mathcal{Z}^\alpha \eta\|_{L^2}^2 d\tau \\ &\leq \frac{1}{2} \int |\mathcal{Z}^\alpha \eta_0|^2 dx + C \mu \varepsilon \int_0^t \|\nabla \eta\|_{\mathcal{H}^{m-2}}^2 d\tau + C_{m+2} \varepsilon \int_0^t \|\eta\|_{\mathcal{H}^{m-1}}^2 d\tau \\ &\quad + \int_0^t \int \mathcal{Z}^\alpha F \cdot \mathcal{Z}^\alpha \eta \, dx d\tau - \mu \varepsilon \int_0^t \int \mathcal{Z}^\alpha (\chi \Delta(\Pi B) \cdot u) \cdot \mathcal{Z}^\alpha \eta \, dx d\tau \\ &\quad + \int_0^t \int (\mathcal{C}_3^\alpha + \mathcal{C}_4^\alpha) \cdot \mathcal{Z}^\alpha \eta \, dx d\tau. \end{aligned} \quad (3.92)$$

Similar to (3.85)-(3.86), we apply the Proposition 2.2 to deduce that

$$\int_0^t \int \mathcal{Z}^\alpha (\chi_{F_1} \times n) \cdot \mathcal{Z}^\alpha \eta dx d\tau \leq C_m \left\{ \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta (1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau \right\}, \quad (3.93)$$

$$\begin{aligned} & \int_0^t \int \mathcal{Z}^\alpha (\chi_{\Pi(BF_2)}) \cdot \mathcal{Z}^\alpha \eta dx d\tau \\ & \leq C_{m+1} \left\{ \delta \int_0^t \|\nabla \partial_t^{m-1} p\|_{L^2}^2 d\tau + \delta \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau + (1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau \right\}, \end{aligned} \quad (3.94)$$

$$\int_0^t \int \mathcal{Z}^\alpha (\chi_{F_3}) \cdot \mathcal{Z}^\alpha \eta dx d\tau \leq C_{m+2} \left\{ \delta \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta (1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau \right\}, \quad (3.95)$$

and

$$\int_0^t \int \mathcal{Z}^\alpha F_4 \cdot \mathcal{Z}^\alpha \eta dx d\tau \leq C_{m+1} \left\{ \delta \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta (1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau \right\}. \quad (3.96)$$

Then, the combination of (3.93)-(3.96) gives directly

$$\begin{aligned} \int_0^t \int \mathcal{Z}^\alpha F \cdot \mathcal{Z}^\alpha \eta dx d\tau & \leq C_{m+2} \left\{ \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau \right\} \\ & + C_{m+2} \left\{ \delta \int_0^t \|\nabla \partial_t^{m-1} p\|_{L^2}^2 d\tau + C_\delta (1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau \right\}. \end{aligned} \quad (3.97)$$

Integrating by parts, one arrives at directly

$$\left| \mu \varepsilon \int_0^t \int \mathcal{Z}^\alpha (\chi_{\Delta(\Pi B)} \cdot u) \cdot \mathcal{Z}^\alpha \eta dx d\tau \right| \leq \delta \mu \varepsilon^2 \int_0^t \|\nabla \eta\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta C_{m+2} \int_0^t \Lambda_m(\tau) d\tau. \quad (3.98)$$

Using the same argument as Lemma 3.13 of [12], one can obtain the following estimates

$$\int_0^t \int (C_3^\alpha + C_4^\alpha) \cdot \mathcal{Z}^\alpha \eta dx d\tau \leq C_m (1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau. \quad (3.99)$$

Substituting (3.97), (3.98) and (3.99) into (3.92), we find

$$\begin{aligned} & \frac{1}{2} \int |\mathcal{Z}^\alpha \eta(t)|^2 dx + \frac{3\mu\varepsilon}{4} \int_0^t \|\nabla \mathcal{Z}^\alpha \eta\|_{L^2}^2 d\tau \\ & \leq C_{m+2} \left\{ \frac{1}{2} \int |\mathcal{Z}^\alpha \eta_0|^2 dx + C\varepsilon \int_0^t \|\nabla \eta\|_{\mathcal{H}^{m-2}}^2 d\tau + \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \right\} \\ & + C_{m+2} \left\{ \delta \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta \int_0^t \|\nabla \partial_t^{m-1} p\|_{L^2}^2 d\tau + C_\delta (1 + P(Q(t))) \int_0^t \Lambda_m(t) d\tau \right\}. \end{aligned}$$

By the induction assumption, one can eliminate the term $\varepsilon \int_0^t \|\nabla \eta\|_{\mathcal{H}^{m-2}}^2 d\tau$. Therefore, we complete the proof of Lemma 3.5. \square

3.3. Estimates for the Δd , $\text{div} u$ and Δp . In this subsection, we shall get some uniform estimates for Δd , $\text{div} u$ and Δp in conormal Sobolev space.

Lemma 3.6. *For a smooth solution to (1.1) and (1.3), it holds that for $\varepsilon \in (0, 1]$*

$$\sup_{0 \leq \tau \leq t} \|\Delta d(\tau)\|_{L^2}^2 + \int_0^t \|\nabla \Delta d\|_{L^2}^2 d\tau \leq \|\Delta d_0\|_{L^2}^2 + C(1 + Q(t)^2) \int_0^t \Lambda_1(\tau) d\tau. \quad (3.100)$$

Proof. Taking ∇ operator to the equation (1.1)₂, one arrives at

$$\nabla d_t - \nabla \Delta d = -\nabla(u \cdot \nabla d) + \nabla(|\nabla d|^2 d). \quad (3.101)$$

Multiplying (3.101) by $-\nabla \Delta d$, we find

$$\begin{aligned} & - \int \nabla d_t \cdot \nabla \Delta d \, dx + \int |\nabla \Delta d|^2 dx \\ & = \int \nabla(u \cdot \nabla d) \cdot \nabla \Delta d \, dx - \int \nabla(|\nabla d|^2 d) \cdot \nabla \Delta d \, dx. \end{aligned} \quad (3.102)$$

By integrating by parts and applying the Neumann boundary condition (1.3), we get

$$- \int \nabla d_t \cdot \nabla \Delta d \, dx = - \int_{\partial\Omega} n \cdot \nabla d_t \cdot \Delta d \, d\sigma + \int \Delta d_t \cdot \Delta d \, dx = \frac{1}{2} \frac{d}{dt} \int |\Delta d|^2 dx. \quad (3.103)$$

In view of the Cauchy inequality, we obtain

$$\begin{aligned} \int \nabla(u \cdot \nabla d) \cdot \nabla \Delta d \, dx & \leq \delta \|\nabla \Delta d\|_{L^2}^2 + C_\delta \|u\|_{W^{1,\infty}}^2 (\|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2), \\ - \int \nabla(|\nabla d|^2 d) \cdot \nabla \Delta d \, dx & \leq \delta \|\nabla \Delta d\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^4 \|\nabla d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2. \end{aligned} \quad (3.104)$$

Substituting (3.104) into (3.103), choosing δ small enough and integrating over $[0, t]$, one attains

$$\begin{aligned} & \frac{1}{2} \int |\Delta d|^2(t) dx + \frac{3}{4} \int |\nabla \Delta d|^2 dx \\ & \leq \int |\Delta d_0|^2 dx + C(\|u\|_{W^{1,\infty}}^2 + \|\nabla d\|_{L^\infty}^4) \int_0^t (\|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) d\tau. \end{aligned}$$

Therefore, we complete the proof of Lemma 3.6. \square

Next, we can establish the following conormal estimates for the quantity Δd .

Lemma 3.7. *For $m \geq 1$ and a smooth solution to (1.1) and (1.3), it holds that for $\varepsilon \in (0, 1]$*

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \|\Delta d(\tau)\|_{\mathcal{H}^{m-1}}^2 + \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & \leq C_m \left\{ \|\Delta d_0\|_{\mathcal{H}^{m-1}}^2 + \delta \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + C_\delta (1 + P(Q(t))) \int_0^t \Lambda_m(t) d\tau \right\}. \end{aligned} \quad (3.105)$$

Proof. The case for $m = 1$ is already proved in Lemma 3.6. Assume that (3.105) is proved for $k = m - 2$. We shall prove that it holds for $k = m - 1 \geq 1$. For $|\alpha| = m - 1$, multiplying (3.101) by $-\nabla \mathcal{Z}^\alpha \Delta d$, we find

$$\begin{aligned} & - \int \mathcal{Z}^\alpha \nabla d_t \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx + \int \mathcal{Z}^\alpha \nabla \Delta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx \\ & = \int \mathcal{Z}^\alpha \nabla(u \cdot \nabla d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx + \int \mathcal{Z}^\alpha \nabla(|\nabla d|^2 d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx. \end{aligned} \quad (3.106)$$

Integrating by part, it is easy to deduce that

$$\begin{aligned} & - \int \mathcal{Z}^\alpha \nabla d_t \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx \\ & = - \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, d\sigma + \int \nabla \cdot (\mathcal{Z}^\alpha \nabla d_t) \cdot \mathcal{Z}^\alpha \Delta d \, dx \\ & = - \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, d\sigma + \frac{1}{2} \frac{d}{dt} \int |\mathcal{Z}^\alpha \Delta d|^2 dx - \int [\mathcal{Z}^\alpha, \nabla \cdot] \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, dx. \end{aligned} \quad (3.107)$$

It is easy to check that

$$\int \mathcal{Z}^\alpha \nabla \Delta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx = \int |\nabla \mathcal{Z}^\alpha \Delta d|^2 dx + \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx. \quad (3.108)$$

Substituting (3.107) and (3.108) into (3.106) and integrating over $[0, t]$, we find

$$\begin{aligned}
& \frac{1}{2} \int |\mathcal{Z}^\alpha \Delta d(t)|^2 dx + \int_0^t \int |\nabla \mathcal{Z}^\alpha \Delta d|^2 dx d\tau \\
&= \frac{1}{2} \int |\mathcal{Z}^\alpha \Delta d_0|^2 dx + \int_0^t \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, d\sigma d\tau + \int_0^t \int [\mathcal{Z}^\alpha, \nabla \cdot] \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, dx d\tau \\
&\quad - \int_0^t \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau + \int_0^t \int \mathcal{Z}^\alpha \nabla (u \cdot \nabla d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\
&\quad + \int_0^t \int \mathcal{Z}^\alpha \nabla (|\nabla d|^2 d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\
&:= III_1 + III_2 + III_3 + III_4 + III_5 + III_6.
\end{aligned} \tag{3.109}$$

To deal with the boundary term on the right hand side of (3.109). If $|\alpha_0| = m - 1$, then we have

$$\int_0^t \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, d\sigma d\tau = 0. \tag{3.110}$$

On the other hand, it is easy to deduce that for $|\alpha_0| \leq m - 2$

$$\int_0^t \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, d\sigma d\tau \leq \int_0^t |n \cdot \mathcal{Z}^\alpha \nabla d_t|_{L^2(\partial\Omega)} |\mathcal{Z}^\alpha \Delta d|_{L^2(\partial\Omega)} d\tau. \tag{3.111}$$

The application of trace inequality in Proposition 2.3 and the boundary condition (1.3) implies

$$\begin{aligned}
|\mathcal{Z}^\alpha \Delta d|_{L^2(\partial\Omega)} &= |\partial_t^{\alpha_0} \Delta d|_{H^{m-1-|\alpha_0|}(\partial\Omega)} \\
&\leq C \|\nabla \partial_t^{\alpha_0} \Delta d\|_{m-1-|\alpha_0|} + C \|\partial_t^{\alpha_0} \Delta d\|_{m-|\alpha_0|} \\
&\leq C \|\nabla \Delta d\|_{\mathcal{H}^{m-1}} + C \|\Delta d\|_{\mathcal{H}^m},
\end{aligned} \tag{3.112}$$

and

$$\begin{aligned}
|n \cdot \mathcal{Z}^\alpha \nabla d_t|_{L^2(\partial\Omega)} &\leq C_m |\partial_t^{\alpha_0} \nabla d_t|_{H^{m-2-|\alpha_0|}(\partial\Omega)} \\
&\leq C_m \|\partial_t^{\alpha_0} \nabla^2 d_t\|_{m-2-|\alpha_0|} + C_m \|\partial_t^{\alpha_0} \nabla d_t\|_{m-1-|\alpha_0|} \\
&\leq C_m \|\nabla^2 d\|_{\mathcal{H}^{m-1}} + C_m \|\nabla d\|_{\mathcal{H}^m}.
\end{aligned} \tag{3.113}$$

Substituting (3.112) and (3.113) into (3.111) and applying the Cauchy inequality, one attains

$$\begin{aligned}
III_2 &\leq \delta_1 \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau \\
&\quad + C_m \left\{ C_{\delta, \delta_1} \int_0^t \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_{\delta, \delta_1} \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau \right\}.
\end{aligned} \tag{3.114}$$

By virtue of the Cauchy inequality, one arrives at

$$\begin{aligned}
III_3 &\leq C \int_0^t \|\Delta d_t\|_{\mathcal{H}^{m-2}} \|\Delta d\|_{\mathcal{H}^{m-1}} d\tau \leq C \int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau, \\
III_4 &\leq \delta_1 \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_{\delta_1} \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-2}}^2 d\tau.
\end{aligned} \tag{3.115}$$

The application of Proposition 2.2 yields directly

$$\begin{aligned}
III_5 &= \int_0^t \int \mathcal{Z}^\alpha (\nabla u \cdot \nabla d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau + \int_0^t \int \mathcal{Z}^\alpha (u \cdot \nabla^2 d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\
&\leq \delta_1 \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_\delta \|\nabla u\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla u\|_{\mathcal{H}^{m-1}}^2 d\tau \\
&\quad + C_\delta \|u\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta \|\nabla^2 d\|_{L_{x,t}^\infty}^2 \int_0^t \|u\|_{\mathcal{H}^{m-1}}^2 d\tau \\
&\leq \delta_1 \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_{\delta_1} (1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau.
\end{aligned} \tag{3.116}$$

It is easy to deduce that

$$\begin{aligned}
III_6 &= \sum_{|\beta| \geq 1} \int_0^t \int \mathcal{Z}^\gamma (\nabla d \cdot \nabla^2 d) \cdot \mathcal{Z}^\beta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\
&\quad + \int_0^t \int \mathcal{Z}^\alpha (\nabla d \cdot \nabla^2 d) \cdot d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\
&\quad + \sum_{|\beta|+|\gamma|=m-1} \int_0^t \int \mathcal{Z}^\gamma (|\nabla d|^2) \mathcal{Z}^\beta \nabla d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\
&= III_{61} + III_{62} + III_{63}.
\end{aligned} \tag{3.117}$$

By virtue of the Proposition 2.2 and Cauchy inequality, one arrives at

$$\begin{aligned}
III_{61} &\leq \delta \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_\delta \|\mathcal{Z} d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d \cdot \nabla^2 d\|_{\mathcal{H}^{m-2}}^2 d\tau \\
&\quad + C_\delta \|\nabla d \cdot \nabla^2 d\|_{L_{x,t}^\infty}^2 \int_0^t \|\mathcal{Z} d\|_{\mathcal{H}^{m-2}}^2 d\tau \\
&\leq \delta_1 \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_1 C_{\delta_1} (1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau.
\end{aligned} \tag{3.118}$$

Similarly, it is easy to deduce that

$$\begin{aligned}
III_{62} &\leq \delta \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_\delta \|\nabla d\|_{W_{x,t}^{1,\infty}}^2 \int_0^t (\|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau, \\
III_{63} &\leq \delta \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_\delta \|\nabla d\|_{L_{x,t}^\infty}^4 \int_0^t (\|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau.
\end{aligned} \tag{3.119}$$

Substituting (3.118) and (3.119) into (3.117), we obtain

$$III_6 \leq \delta \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_1 C_\delta (1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau. \tag{3.120}$$

Substituting (3.114)-(3.116) and (3.120) into (3.109) and choosing δ_1 small enough, we find

$$\begin{aligned}
&\frac{1}{2} \int |\mathcal{Z}^\alpha \Delta d(t)|^2 dx + \int_0^t \int |\nabla \mathcal{Z}^\alpha \Delta d|^2 dx d\tau \\
&\leq C_m \left\{ \frac{1}{2} \int |\mathcal{Z}^\alpha \Delta d_0|^2 dx + \delta \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + C \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-2}}^2 d\tau \right\} \\
&\quad + C_m C_\delta [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau.
\end{aligned}$$

By the induction assumption, one can eliminate the term $\int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-2}}^2 d\tau$. Therefore, we complete the proof of Lemma 3.7. \square

Next, we derive the following lower order estimates on $\|(\operatorname{div} u, p)\|_{L^2}^2$.

Lemma 3.8. *For every $m \in \mathbb{N}_+$, it holds that*

$$\begin{aligned}
&\sup_{0 \leq \tau \leq t} \int \left(\frac{1}{2} \rho |\operatorname{div} u|^2 + \frac{1}{2\gamma p} |\nabla p(\tau)|^2 \right) dx + \varepsilon \int_0^t \|\nabla \operatorname{div} u(\tau)\|_{L^2}^2 d\tau \\
&\leq \int \left(\frac{1}{2} \rho_0 |\operatorname{div} u_0|^2 + \frac{1}{2\gamma p_0} |\nabla p_0|^2 \right) dx + C_3 [1 + P(Q(t))] \int_0^t (\Lambda_m(\tau) + \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau.
\end{aligned} \tag{3.121}$$

Proof. Multiplying (3.5) by $\nabla \operatorname{div} u$ yields that

$$\begin{aligned} & \int_0^t \int (\rho u_t + \rho u \cdot \nabla u) \cdot \nabla \operatorname{div} u \, dx d\tau + \int_0^t \int \nabla p \cdot \nabla \operatorname{div} u \, dx d\tau \\ &= -\mu \varepsilon \int_0^t \int \nabla \times w \cdot \nabla \operatorname{div} u \, dx d\tau + (2\mu + \lambda) \varepsilon \int_0^t \int |\nabla \operatorname{div} u|^2 \, dx d\tau \\ & \quad - \int_0^t \int (\nabla d \cdot \Delta d) \cdot \nabla \operatorname{div} u \, dx d\tau = IV_1 + IV_2 + IV_3 + IV_4. \end{aligned} \quad (3.122)$$

Using the same argument as Lemma 3.5 of [12], one can obtain the following estimates

$$IV_1 \leq -\frac{1}{2} \int \rho |\operatorname{div} u|^2 \, dx + \frac{1}{2} \int \rho_0 |\operatorname{div} u_0|^2 \, dx + C_2 [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) \, d\tau, \quad (3.123)$$

$$IV_2 \leq - \int \frac{1}{2\gamma p} |\nabla p|^2 \, dx + \int \frac{1}{2\gamma p_0} |\nabla p_0|^2 \, dx + C_2 [1 + P(Q(t))] \int_0^t \|\nabla p\|_{L^2}^2 \, d\tau \quad (3.124)$$

and

$$IV_3 \leq \frac{\varepsilon}{4} \int_0^t \|\nabla \operatorname{div} u\|_{L^2}^2 \, d\tau + C_3 \varepsilon \int_0^t (\|u(\tau)\|_{L^2}^2 + \|\nabla u(\tau)\|_{L^2}^2) \, d\tau. \quad (3.125)$$

Integrating by part and applying the boundary condition (1.3), we find

$$\begin{aligned} IV_4 &= - \int_0^t \int_{\partial\Omega} n \cdot (\nabla d \cdot \Delta d) \operatorname{div} u \, d\sigma d\tau + \int_0^t \int \nabla (\nabla d \cdot \Delta d) \operatorname{div} u \, dx d\tau \\ &\leq C (\|\nabla d\|_{L^\infty}^2 + \|\Delta d\|_{L^\infty}) \int_0^t (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) \, d\tau. \end{aligned} \quad (3.126)$$

Substituting (3.123)-(3.126) into (3.122), we find

$$\begin{aligned} & \int \left(\frac{1}{2} \rho |\operatorname{div} u|^2 + \frac{1}{2\gamma p} |\nabla p|^2 \right) \, dx + \varepsilon \int_0^t \|\nabla \operatorname{div} u(\tau)\|_{L^2}^2 \, d\tau \\ &\leq \int \left(\frac{1}{2} \rho_0 |\operatorname{div} u_0|^2 + \frac{1}{2\gamma p_0} |\nabla p_0|^2 \right) \, dx + C_3 [1 + P(Q(t))] \int_0^t (\Lambda_m(\tau) + \|\nabla \Delta d\|_{L^2}^2) \, d\tau. \end{aligned}$$

Therefore, we complete the proof of Lemma 3.8. \square

Lemma 3.9. For $m \geq 1$ and $|\alpha| \leq m-1$ with $|\alpha_0| \leq m-2$, it holds that

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \int \left(\rho |\mathcal{Z}^\alpha \operatorname{div} u(\tau)|^2 + \frac{1}{\gamma p} |\mathcal{Z}^\alpha \nabla p(\tau)|^2 \right) \, dx + \varepsilon \int_0^t \|\nabla \mathcal{Z}^\alpha \operatorname{div} u\|_{L^2}^2 \, d\tau \\ &\leq C \int \left(\rho_0 |\mathcal{Z}^\alpha \operatorname{div} u_0|^2 + \frac{1}{\gamma p_0} |\mathcal{Z}^\alpha \nabla p_0|^2 \right) \, dx + C C_{m+2} \delta \int_0^t \|\nabla \partial_t^{m-1} p(\tau)\|_{L^2}^2 \, d\tau \\ & \quad + C C_{m+2} \left\{ \varepsilon \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-2}}^2 \, d\tau + (\delta + \varepsilon) \int_0^t \|\nabla \mathcal{Z}^{m-2} \operatorname{div} u\|_{L^2}^2 \, d\tau \right\} \\ & \quad + C_\delta C_{m+2} [1 + P(Q(t))] \int_0^t (\Lambda_m + \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2) \, d\tau. \end{aligned} \quad (3.127)$$

Proof. The case for $|\alpha| = 0$ is already proved in Lemma 3.8. Assuming it is proved for $|\alpha| \leq m-2$, one needs to prove it for $|\alpha| = m-1$ with $|\alpha_0| \leq m-2$. Multiplying (3.5) by $\nabla \mathcal{Z}^\alpha \operatorname{div} u$ leads to

$$\begin{aligned}
& \underbrace{\int_0^t \int (\rho \mathcal{Z}^\alpha u_t + \rho u \cdot \nabla \mathcal{Z}^\alpha u) \cdot \nabla \mathcal{Z}^\alpha \operatorname{div} u \, dx d\tau}_{V_1} + \underbrace{\int_0^t \int \mathcal{Z}^\alpha \nabla p \cdot \nabla \mathcal{Z}^\alpha \operatorname{div} u \, dx d\tau}_{V_2} \\
&= \underbrace{-\mu \varepsilon \int_0^t \int \mathcal{Z}^\alpha \nabla \times w \cdot \nabla \mathcal{Z}^\alpha \operatorname{div} u \, dx d\tau}_{V_3} + \underbrace{(2\mu + \lambda) \varepsilon \int_0^t \int \mathcal{Z}^\alpha \nabla \operatorname{div} u \cdot \nabla \mathcal{Z}^\alpha \operatorname{div} u \, dx d\tau}_{V_4} \\
&\quad - \underbrace{\int_0^t \int \mathcal{Z}^\alpha (\nabla d \cdot \Delta d) \cdot \nabla \mathcal{Z}^\alpha \operatorname{div} u \, dx d\tau}_{V_5} + \underbrace{\int_0^t \int (\mathcal{C}_1^\alpha + \mathcal{C}_2^\alpha) \cdot \nabla \mathcal{Z}^\alpha \operatorname{div} u \, dx d\tau}_{V_6}.
\end{aligned} \tag{3.128}$$

Using the same argument as Lemma 3.6 of [12], one can obtain the following estimates

$$\begin{aligned}
V_1 &\leq -\int \frac{\rho}{2} |\mathcal{Z}^\alpha \operatorname{div} u|^2 dx + \int \frac{\rho_0}{2} |\mathcal{Z}^\alpha \operatorname{div} u_0|^2 dx + \delta \int_0^t \|\nabla \mathcal{Z}^{\alpha-2} \operatorname{div} u\|_{L^2}^2 d\tau \\
&\quad + C_\delta [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau,
\end{aligned} \tag{3.129}$$

$$V_4 \geq \frac{3(2\mu + \lambda)\varepsilon}{4} \int_0^t \|\nabla \mathcal{Z}^\alpha \operatorname{div} u\|_{L^2}^2 d\tau - C\varepsilon \int_0^t \|\nabla \mathcal{Z}^{m-2} \operatorname{div} u\|_{L^2}^2 d\tau - C\varepsilon \int_0^t \Lambda_m(\tau) d\tau, \tag{3.130}$$

$$V_3 \geq -\frac{(2\mu + \lambda)\varepsilon}{4} \int_0^t \|\nabla \mathcal{Z}^\alpha \operatorname{div} u\|_{L^2}^2 d\tau - C\varepsilon \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-2}}^2 d\tau - C_{m+2} \int_0^t P(\Lambda_m(\tau)) d\tau, \tag{3.131}$$

$$\begin{aligned}
V_2 &\leq -\int \frac{1}{2\gamma p} |\mathcal{Z}^\alpha \nabla p|^2 dx + \int \frac{1}{2\gamma p_0} |\mathcal{Z}^\alpha \nabla p_0|^2 dx + C\delta \int_0^t \|\nabla \mathcal{Z}^{m-2} \operatorname{div} u\|_{L^2}^2 d\tau \\
&\quad + C\delta \int_0^t \|\nabla \partial_t^{m-1} p\|_{L^2}^2 d\tau + C_\delta C_{m+1} (1 + P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau,
\end{aligned} \tag{3.132}$$

and

$$V_6 \leq \delta \int_0^t \|\nabla \mathcal{Z}^{m-2} \operatorname{div} u\|_{L^2}^2 d\tau + C_\delta [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \tag{3.133}$$

On the other hand, the integration by parts yields directly

$$V_5 = -\int_0^t \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha (\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha \operatorname{div} u \, d\sigma d\tau + \int_0^t \int \operatorname{div} \mathcal{Z}^\alpha (\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha \operatorname{div} u \, dx d\tau. \tag{3.134}$$

In view of the trace inequality in Proposition 2.3, we find

$$|Z_y^{m-2-\alpha_0} Z_t^{\alpha_0} \operatorname{div} u|_{L^2(\partial\Omega)}^2 \leq C \|\nabla Z_y^{m-2-\alpha_0} Z_t^{\alpha_0} \operatorname{div} u\|_{L^2}^2 + C \|Z_y^{m-2-\alpha_0} Z_t^{\alpha_0} \operatorname{div} u\|_{L^2}^2, \tag{3.135}$$

and

$$|Z_y(n \cdot Z_y^{m-1-\alpha_0} Z_t^{\alpha_0})(\nabla d \cdot \Delta d)|_{L^2(\partial\Omega)}^2 \leq C \|\nabla Z_t^{\alpha_0} (\nabla d \cdot \Delta d)\|_{H_{co}^{|\alpha_1|}}^2 + C \|Z_t^{\alpha_0} (\nabla d \cdot \Delta d)\|_{H_{co}^{|\alpha_1|}}^2. \tag{3.136}$$

Integrating by parts along the boundary and applying the estimates (3.135) and (3.136), one attains

$$\begin{aligned}
& - \int_0^t \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha (\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha \operatorname{div} u \, d\sigma d\tau \\
& \leq \int_0^t |Z_y(n \cdot Z_y^{m-1-\alpha_0} Z_t^{\alpha_0}) (\nabla d \cdot \Delta d)|_{L^2(\partial\Omega)} |Z_y^{m-2-\alpha_0} Z_t^{\alpha_0} \operatorname{div} u|_{L^2(\partial\Omega)} d\tau \\
& \leq \delta \int_0^t \|\nabla \mathcal{Z}^{m-2} \operatorname{div} u\|_{L^2}^2 d\tau + C_\delta \int_0^t \|\mathcal{Z}^{m-2} \operatorname{div} u\|_{L^2}^2 d\tau + C_\delta \|\nabla^2 d\|_{L^\infty}^2 \int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\
& \quad + C_\delta \|\nabla d\|_{L^\infty}^2 \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta \|\nabla \Delta d\|_{L^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau \\
& \quad + C_\delta \|\nabla d\|_{L^\infty}^2 \int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta \|\Delta d\|_{L^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau \\
& \leq \delta \int_0^t \|\nabla \mathcal{Z}^{m-2} \operatorname{div} u\|_{L^2}^2 d\tau + C_\delta [1 + P(Q(t))] \int_0^t (\Lambda_m + \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau.
\end{aligned} \tag{3.137}$$

Applying the Proposition 2.2, it is easy to check that

$$\int_0^t \int \operatorname{div} \mathcal{Z}^\alpha (\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha \operatorname{div} u \, dx d\tau \leq C[1 + P(Q(t))] \int_0^t (\Lambda_m + \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau. \tag{3.138}$$

Substituting (3.137) and (3.138) into (3.136), we obtain

$$V_5 \leq \delta \int_0^t \|\nabla \mathcal{Z}^{m-2} \operatorname{div} u\|_{L^2}^2 d\tau + C_\delta [1 + P(Q(t))] \int_0^t (\Lambda_m + \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau \tag{3.139}$$

Substituting (3.129)-(3.133) and (3.139) into (3.128), we complete the proof of Lemma 3.9. \square

Lemma 3.10. *For $m \geq 1$, it holds that*

$$\begin{aligned}
& \sup_{0 \leq \tau \leq t} \varepsilon \int \left(\rho |\partial_t^{m-1} \operatorname{div} u(\tau)|^2 + \frac{1}{\gamma p} |\partial_t^{m-1} \nabla p(\tau)|^2 \right) dx + \varepsilon^2 \int_0^t \|\nabla \partial_t^{m-1} \operatorname{div} u(\tau)\|_{L^2}^2 d\tau \\
& \leq C\varepsilon \int \left(\rho_0 |\partial_t^{m-1} \operatorname{div} u_0|^2 + \frac{1}{\gamma p_0} |\partial_t^{m-1} \nabla p_0|^2 \right) dx \\
& \quad + C_{m+1} [1 + P(Q(t))] \int_0^t (\Lambda_m(\tau) + \|\nabla \Delta d(\tau)\|_{\mathcal{H}^{m-1}}^2) d\tau.
\end{aligned} \tag{3.140}$$

Proof. Applying ∂_t^{m-1} to equation (3.5), we find that

$$\begin{aligned}
& \rho \partial_t^{m-1} u_t + \rho u \cdot \nabla \partial_t^{m-1} u + \mu \varepsilon \nabla \times \partial_t^{m-1} (\nabla \times u) + \partial_t^{m-1} \nabla p \\
& = (2\mu + \lambda) \varepsilon \nabla \partial_t^{m-1} \operatorname{div} u - \partial_t^{m-1} (\nabla d \cdot \Delta d) + \mathcal{C}_1^{m-1} + \mathcal{C}_2^{m-1},
\end{aligned} \tag{3.141}$$

where

$$\mathcal{C}_1^{m-1} \triangleq -[\partial_t^{m-1}, \rho] u_t, \quad \mathcal{C}_2^{m-1} \triangleq -[\partial_t^{m-1}, \rho u \cdot \nabla] u.$$

Multiplying (3.141) by $\varepsilon \nabla \operatorname{div} \partial_t^{m-1} u$, it is easy to deduce that

$$\begin{aligned}
& \underbrace{\varepsilon \int_0^t \int (\rho \partial_t^{m-1} u_t + \rho u \cdot \nabla \partial_t^{m-1} u) \cdot \nabla \operatorname{div} \partial_t^{m-1} u \, dx d\tau}_{VI_1} + \underbrace{\varepsilon \int_0^t \int \partial_t^{m-1} \nabla p \cdot \nabla \operatorname{div} \partial_t^{m-1} u \, dx d\tau}_{VI_2} \\
& = \underbrace{-\mu \varepsilon^2 \int_0^t \int \nabla \times \partial_t^{m-1} (\nabla \times u) \cdot \nabla \operatorname{div} \partial_t^{m-1} u \, dx d\tau}_{VI_3} + \underbrace{(2\mu + \lambda) \varepsilon^2 \int_0^t \int |\nabla \partial_t^{m-1} \operatorname{div} u|^2 dx d\tau}_{VI_4} \\
& \quad + \underbrace{\varepsilon \int_0^t \int (\mathcal{C}_1^{m-1} + \mathcal{C}_2^{m-1}) \cdot \nabla \operatorname{div} \partial_t^{m-1} u \, dx d\tau}_{VI_5} - \underbrace{\varepsilon \int_0^t \int \partial_t^{m-1} (\nabla d \cdot \Delta d) \cdot \nabla \operatorname{div} \partial_t^{m-1} u \, dx d\tau}_{VI_6}.
\end{aligned} \tag{3.142}$$

Using the same argument as Lemma 3.8 of [12], one can obtain the following estimates

$$|VI_3| \leq \frac{2\mu + \lambda}{8} \varepsilon^4 \int_0^t \|\nabla \partial_t^{m-1} \operatorname{div} u\|_{L^2}^2 d\tau + CC_3 \int_0^t \Lambda_m(\tau) d\tau. \quad (3.143)$$

$$\begin{aligned} VI_1 &\leq -\varepsilon \int \rho |\partial_t^{m-1} \operatorname{div} u(t)|^2 dx + \varepsilon \int \rho_0 |\partial_t^{m-1} \operatorname{div} u_0|^2 dx \\ &\quad + \frac{\varepsilon^2}{8} \int_0^t \|\nabla \partial_t^{m-1} \operatorname{div} u\|_{L^2}^2 d\tau + C[1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau, \end{aligned} \quad (3.144)$$

$$|VI_5| \leq \frac{2\mu + \lambda}{8} \varepsilon^2 \int_0^t \|\nabla \partial_t^{m-1} \operatorname{div} u\|_{L^2}^2 d\tau + C[1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \quad (3.145)$$

and

$$VI_2 \leq -\varepsilon \int \frac{1}{2\gamma p} |\nabla \partial_t^{m-1} p|^2 dx + \varepsilon \int \frac{1}{2\gamma p_0} |\nabla \partial_t^{m-1} p_0|^2 dx + C_{m+1} [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \quad (3.146)$$

On the other hand, integrating by parts and applying the boundary condition (1.3), one attains

$$\begin{aligned} VI_6 &= -\varepsilon \int_0^t \int n \cdot \partial_t^{m-1} (\nabla d \cdot \Delta d) \cdot \operatorname{div} \partial_t^{m-1} u \, d\sigma d\tau \\ &\quad + \varepsilon \int_0^t \int \operatorname{div} \partial_t^{m-1} (\nabla d \cdot \Delta d) \cdot \operatorname{div} \partial_t^{m-1} u \, dx d\tau \\ &\leq C[1 + P(Q(t))] \int_0^t (\Lambda_m(\tau) + \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2) \tau. \end{aligned} \quad (3.147)$$

Substituting (3.143)-(3.147) into (3.142), we complete the proof of Lemma 3.10. \square

Next, we recall an important estimate that has been proved by Wang et al. [12].

Lemma 3.11. *Define*

$$\Lambda_{1m}(t) \triangleq \|(p, u, \nabla d)(t)\|_{\mathcal{H}^m}^2 + \|\Delta d(t)\|_{\mathcal{H}^{m-1}}^2 + \sum_{|\beta| \leq m-2} \|\mathcal{Z}^\beta \nabla p(t)\|_1^2 + \sum_{|\beta| \leq m-2} \|\mathcal{Z}^\beta \nabla u(t)\|_1^2. \quad (3.148)$$

Then, for every $m \geq 3$, it holds that

$$\|\partial_t^{m-1} \operatorname{div} u(t)\|_{L^2}^2 \leq C_2 \{P(\Lambda_{1m}(t)) + P(Q(t))\}. \quad (3.149)$$

Lemma 3.12. *For every $m \geq 1$, it holds that*

$$\int_0^t \|\nabla \partial_t^{m-1} p(\tau)\|_{L^2}^2 d\tau \leq C\varepsilon^2 \int_0^t \|\nabla^2 \partial_t^{m-1} u(\tau)\|_{L^2}^2 d\tau + C[1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \quad (3.150)$$

Proof. Applying ∂_t^{m-1} to (3.5), we find

$$\partial_t^{m-1} \nabla p = \partial_t^{m-1} (-\rho u_t - \rho u \cdot \nabla u - \mu \varepsilon \nabla \times (\nabla \times u) + (2\mu + \lambda) \varepsilon \nabla \operatorname{div} u - \nabla d \cdot \Delta d).$$

By using the Proposition 2.2, it is easy to deduce the estimate (3.150). Hence, we complete the proof of Lemma 3.12. \square

Next, we recall an important estimate that has been proved by Wang et al. [12].

Lemma 3.13. *For every $m \geq 1$, it holds that*

$$\int_0^t \|\nabla \mathcal{Z}^{m-2} \operatorname{div} u(\tau)\|_{L^2}^2 d\tau \leq C \int_0^t \|\nabla \partial_t^{m-1} p(\tau)\|_{L^2}^2 d\tau + C_m [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \quad (3.151)$$

Finally, we estimate the estimate for the quantity $\nabla \Delta d$.

Lemma 3.14. *For every $m \geq 1$, it holds that*

$$\int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \leq C[1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \quad (3.152)$$

Proof. By virtue of the equation (1.1)₃, it is easy to deduce that

$$\nabla \Delta d = \nabla d_t + \nabla(u \cdot \nabla d) - \nabla(|\nabla d|^2 d).$$

By using the Proposition 2.2, it is easy to deduce the estimate (3.152). Then, we complete the proof of Lemma 3.14. \square

Substituting (3.150), (3.151) and (3.152) into (3.127), it is easy to deduce that

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \sum_{k=0}^{m-2} \|(\partial_t^k \nabla p, \partial_t^k \operatorname{div} u)(\tau)\|_{m-1-k}^2 + \varepsilon \int_0^t \sum_{k=0}^{m-2} \|\partial_t^k \nabla \operatorname{div} u(\tau)\|_{m-1-k}^2 d\tau \\ & \leq CC_{m+2} \left\{ \Lambda_m(0) + C\delta\varepsilon^2 \int_0^t \|\nabla^2 \partial_t^{m-1} u(\tau)\|_{L^2}^2 d\tau + \varepsilon \int_0^t \|\nabla^2 u(\tau)\|_{\mathcal{H}^{m-2}}^2 d\tau \right\} \\ & \quad + C_\delta C_{m+2} [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \end{aligned} \quad (3.153)$$

Substituting (3.152) into (3.140), we obtain

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \varepsilon (\|\partial_t^{m-1} \operatorname{div} u(\tau)\|_{L^2}^2 + \|\partial_t^{m-1} \nabla p(\tau)\|_{L^2}^2) + \varepsilon^2 \int_0^t \|\nabla \partial_t^{m-1} \operatorname{div} u(\tau)\|_{L^2}^2 d\tau \\ & \leq C\varepsilon \Lambda_m(0) + CC_{m+1} [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \end{aligned} \quad (3.154)$$

Then, the combination of (3.153) and (3.154) yields directly

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \sum_{k=0}^{m-2} \left\{ \|(\partial_t^k \nabla p, \partial_t^k \operatorname{div} u)(\tau)\|_{m-1-k}^2 + \varepsilon \|(\partial_t^{m-1} \operatorname{div} u, \partial_t^{m-1} \nabla p)(\tau)\|_{L^2}^2 \right\} \\ & + \varepsilon \int_0^t \sum_{k=0}^{m-2} \|\partial_t^k \nabla \operatorname{div} u(\tau)\|_{m-1-k}^2 d\tau + \varepsilon^2 \int_0^t \|\nabla \partial_t^{m-1} \operatorname{div} u(\tau)\|_{L^2}^2 d\tau \\ & \leq CC_{m+2} \left\{ \Lambda_m(0) + C\delta\varepsilon^2 \int_0^t \|\nabla^2 \partial_t^{m-1} u(\tau)\|_{L^2}^2 d\tau + \varepsilon \int_0^t \|\nabla^2 u(\tau)\|_{\mathcal{H}^{m-2}}^2 d\tau \right\} \\ & \quad + C_\delta C_{m+2} [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \end{aligned} \quad (3.155)$$

On the other hand, it is easy to check that

$$\sum_{|\beta| \leq m-2} \|\mathcal{Z}^\beta \nabla u\|_1^2 \leq C_{m+1} (\|u\|_{\mathcal{H}^m}^2 + \|\eta\|_{\mathcal{H}^{m-1}}^2 + \sum_{k=0}^{m-2} \|\partial_t^k \operatorname{div} u\|_{m-1-k}^2), \quad (3.156)$$

$$\int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau \leq C_{m+2} \int_0^t (\|\nabla u\|_{\mathcal{H}^m}^2 + \|\nabla \eta\|_{\mathcal{H}^{m-1}}^2 + \|\nabla \operatorname{div} u\|_{\mathcal{H}^{m-1}}^2 + \Lambda_m) d\tau, \quad (3.157)$$

$$\varepsilon \int_0^t \|\nabla^2 \mathcal{Z}^{m-2} u\|_{L^2}^2 d\tau \leq C_{m+1} \varepsilon \int_0^t \|\nabla \eta\|_{\mathcal{H}^{m-2}}^2 d\tau + C_{m+1} \int_0^t \Lambda_m(\tau) d\tau, \quad (3.158)$$

and

$$\begin{aligned} & \sum_{k=0}^{m-2} \int_0^t \|\nabla^2 \partial_t^k u\|_{m-1-k}^2 d\tau \\ & \leq C_{m+2} \int_0^t (\|\nabla u\|_{\mathcal{H}^m}^2 + \|\nabla \eta\|_{\mathcal{H}^{m-1}}^2) d\tau + C_{m+2} \int_0^t \Lambda_m(\tau) d\tau \\ & \quad + C_{m+2} \sum_{k=0}^{m-2} \int_0^t \|\partial_t^k \nabla \operatorname{div} u\|_{m-1-k}^2 d\tau. \end{aligned} \quad (3.159)$$

The combination of (3.155)-(3.159) yields directly

$$\begin{aligned}
& \sup_{0 \leq \tau \leq t} \{ \Lambda_{1m}(\tau) + \|(\eta, \Delta d)(\tau)\|_{\mathcal{H}^{m-1}}^2 + \varepsilon \|(\partial_t^{m-1} \operatorname{div} u, \partial_t^{m-1} \nabla p)(\tau)\|_{L^2}^2 \} \\
& + \varepsilon \int_0^t (\|\nabla u\|_{\mathcal{H}^m}^2 + \|\nabla \eta\|_{\mathcal{H}^{m-1}}^2) d\tau + \int_0^t (\|\Delta d\|_{\mathcal{H}^m}^2 + \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau \\
& + \varepsilon \int_0^t \sum_{k=0}^{m-2} \|\partial_t^k \nabla \operatorname{div} u\|_{m-1-k}^2 d\tau + \varepsilon^2 \int_0^t \|\partial_t^{m-1} \nabla \operatorname{div} u\|_{L^2}^2 d\tau \\
& + \varepsilon \int_0^t \sum_{k=0}^{m-2} \|\partial_t^k \nabla^2 u\|_{m-1-k}^2 d\tau + \varepsilon^2 \int_0^t \|\partial_t^{m-1} \nabla^2 u\|_{L^2}^2 d\tau + \int_0^t \|\partial_t^{m-1} \nabla p\|_{L^2}^2 d\tau \\
& \leq CC_{m+2} \left\{ \Lambda_m(0) + [1 + P(Q(t))] \int_0^t P(\Lambda_m(\tau)) d\tau \right\}.
\end{aligned} \tag{3.160}$$

3.4. L^∞ - estimates. In this subsection, we shall provide the L^∞ -estimates of (p, u, d) which are needed to estimate on the right-hand side of the estimate (3.160).

Lemma 3.15. *For a smooth solution (p, u, d) to (1.1) and (1.3), it holds that*

$$\|\mathcal{Z}^\alpha(\ln \rho, p, u)\|_{L^\infty}^2 \leq CP(\Lambda_{1m}(t)), \quad m \geq 2 + |\alpha|, \tag{3.161}$$

$$\|\nabla(\ln \rho, p)\|_{\mathcal{H}^{1,\infty}}^2 \leq C_3 (P(\|\Delta p\|_{\mathcal{H}^1}^2) + P(\Lambda_{1m}(t))), \quad m \geq 5, \tag{3.162}$$

$$\|\operatorname{div} u(t)\|_{\mathcal{H}^{1,\infty}}^2 \leq C_3 (P(\|\Delta p\|_{\mathcal{H}^1}^2) + P(\Lambda_{1m}(t))), \quad m \geq 5, \tag{3.163}$$

$$\|\nabla \operatorname{div} u(t)\|_{L^\infty}^2 \leq C_3 P(Q(t)), \tag{3.164}$$

$$\|\nabla \operatorname{div} u(t)\|_{\mathcal{H}^{1,\infty}}^2 \leq C_4 [1 + P(Q(t))] (\delta \|\Delta p\|_{\mathcal{H}^2}^2 + C_\delta P(\Lambda_{1m}(t))), \tag{3.165}$$

$$\|d_t\|_{W^{1,\infty}}^2 + \|\nabla d\|_{\mathcal{H}^{1,\infty}}^2 + \|\nabla^2 d\|_{\mathcal{H}^{1,\infty}}^2 + \|\nabla \Delta d\|_{\mathcal{H}^{1,\infty}}^2 \leq C_{m+2} P(\Lambda_m(t)), \quad m \geq 3. \tag{3.166}$$

Proof. The estimates (3.161)-(3.165) have been proven by Wang et al.[12](see Lemma 3.14). Hence, we give the proof for the estimate (3.166). By virtue of the Sobolev inequality in Proposition 2.3, one arrives at

$$\|\nabla d\|_{L^\infty}^2 \leq C(\|\nabla^2 d\|_1^2 + \|\nabla d\|_2^2). \tag{3.167}$$

In view of the standard elliptic regularity results with Neumann boundary condition, we get that

$$\|\nabla^2 d\|_m^2 \leq C_{m+2}(\|\Delta d\|_m^2 + \|\nabla d\|_{L^2}^2). \tag{3.168}$$

Then, the combination of (3.167) and (3.168) yields directly

$$\|\nabla d\|_{L^\infty}^2 \leq C_3(\|\Delta d\|_1^2 + \|\nabla d\|_2^2). \tag{3.169}$$

For $|\alpha| = 1$, the application of Proposition 2.3 gives for $m \geq 3$

$$\|\mathcal{Z}^\alpha \nabla d\|_{L^\infty}^2 \leq C(\|\nabla(\mathcal{Z}^\alpha \nabla d)\|_1 + \|\mathcal{Z}^\alpha \nabla d\|_1) \|\mathcal{Z}^\alpha \nabla d\|_2 \leq C_{m+2} P(\Lambda_m(t)),$$

which, together with (3.169), yields

$$\|\nabla d\|_{\mathcal{H}^{1,\infty}}^2 \leq C_{m+2} P(\Lambda_m(t)), \quad \text{for } m \geq 3. \tag{3.170}$$

By virtue of the equation (1.1)₃, we find

$$\begin{aligned}
\|d_t\|_{L^\infty}^2 & \leq C(\|\nabla d_t\|_1^2 + \|d_t\|_2^2) \\
& \leq C(\|\nabla d_t\|_1^2 + \|\Delta d\|_2^2 + \|u \cdot \nabla d\|_2^2 + \|\nabla d\|^2 d\|_2^2).
\end{aligned} \tag{3.171}$$

By Proposition 2.2, (3.161) and (3.169), one attains

$$\|u \cdot \nabla d\|_2^2 \leq C(\|u\|_{L^\infty}^2 \|\nabla d\|_2^2 + \|\nabla d\|_{L^\infty}^2 \|u\|_2^2) \leq C_3 \Lambda_m(t), \quad \text{for } m \geq 2; \tag{3.172}$$

and

$$\begin{aligned}
\| |\nabla d|^2 d \|_2^2 &\leq \sum_{|\gamma| \geq 1, |\beta| + |\gamma| \leq 2} \int |Z^\beta (|\nabla d|^2) Z^\gamma d|^2 dx + \| |\nabla d|^2 \|_2^2 \\
&\leq \|Zd\|_{L^\infty}^2 \| |\nabla d|^2 \|_1^2 + \| |\nabla d|^2 \|_{L^\infty}^2 \|Zd\|_1^2 + \| \nabla d \|_{L^\infty}^2 \| \nabla d \|_2^2 \\
&\leq C_3 \Lambda_m^3(t), \text{ for } m \geq 3.
\end{aligned} \tag{3.173}$$

Then the combination of (3.172) and (3.173) gives directly

$$\|d_t\|_{L^\infty}^2 \leq C_3 P(\Lambda_m(t)), \text{ for } m \geq 3. \tag{3.174}$$

By virtue of Proposition 2.3, we obtain for $m \geq 3$

$$\| \nabla d_t \|_{L^\infty}^2 \leq C(\| \nabla^2 d_t \|_1^2 + \| \nabla d_t \|_2^2) \leq C(\| \Delta d_t \|_1^2 + \| \nabla d_t \|_2^2) \leq C(\| \Delta d \|_{\mathcal{H}^2}^2 + \| \nabla d \|_{\mathcal{H}^3}^2),$$

which, together with (3.174), yields immediately

$$\|d_t\|_{W^{1,\infty}}^2 \leq C_3 P(\Lambda_m(t)), \text{ for } m \geq 3. \tag{3.175}$$

On the other hand, it is easy to check that

$$\begin{aligned}
\partial_{ii} &= \partial_{y_i}^2 - \partial_{y_i}(\partial_i \psi \partial_z) - \partial_i \psi \partial_z \partial_{y_i} + (\partial_i \psi)^2 \partial_z^2, \quad i = 1, 2, \\
\partial_1 \partial_2 &= \partial_{y_1} \partial_{y_2} - \partial_{y_2}(\partial_1 \psi \partial_z) - \partial_2 \psi \partial_{y_1} \partial_z + \partial_2 \psi \partial_1 \psi \partial_z^2, \\
\partial_i \partial_3 &= \partial_{y_i} \partial_z - \partial_i \psi \partial_z^2, \quad i = 1, 2.
\end{aligned}$$

Then, we find that

$$\Delta = (1 + |\nabla \psi|^2) \partial_z^2 + \sum_{i=1,2} (\partial_{y_i}^2 - \partial_{y_i}(\partial_i \psi \partial_z) - \partial_i \psi \partial_z \partial_{y_i}). \tag{3.176}$$

and

$$\begin{aligned}
\nabla^2 &= [(1 + |\nabla \psi|^2) + \partial_2 \psi \partial_1 \psi - \partial_1 \psi - \partial_2 \psi] \partial_z^2 + \partial_{y_1} \partial_{y_2} \\
&\quad + \sum_{i=1,2} (\partial_{y_i}^2 - \partial_{y_i}(\partial_i \psi \partial_z) - \partial_i \psi \partial_z \partial_{y_i}) - \partial_{y_2}(\partial_1 \psi \partial_z) \\
&\quad - \partial_2 \psi \partial_{y_1} \partial_z + \partial_{y_1} \partial_z + \partial_{y_2} \partial_z.
\end{aligned} \tag{3.177}$$

The combination of (3.176)-(3.177) and Proposition 2.3 yield that

$$\begin{aligned}
\| \nabla^2 d \|_{L^\infty}^2 &\leq C_1 (\| \Delta d \|_{L^\infty}^2 + \| \partial_z \partial_{y_i} d \|_{L^\infty}^2 + \| \partial_{y_i} \partial_{y_j} d \|_{L^\infty}^2) \\
&\leq C_1 (\| \nabla \Delta d \|_1^2 + \| \Delta d \|_2^2 + \| \nabla \partial_z \partial_{y_i} d \|_1^2 + \| \partial_z \partial_{y_i} d \|_2^2) \\
&\quad + C(\| \nabla \partial_{y_i} \partial_{y_j} d \|_1^2 + \| \partial_{y_i} \partial_{y_j} d \|_2^2) \\
&\leq C_1 (\| \nabla \Delta d \|_1^2 + \| \Delta d \|_2^2 + \| \nabla^2 d \|_2^2 + \| \nabla d \|_3^2) \\
&\leq C_4 (\| \nabla \Delta d \|_1^2 + \| \Delta d \|_2^2 + \| \nabla d \|_3^2),
\end{aligned} \tag{3.178}$$

where we have used the estimate (3.168) in the last inequality. In order to deal with the first term on the right hand side of (3.178), we apply the equation (1.1)₃ to attain that

$$\begin{aligned}
\| \nabla \Delta d \|_1^2 &\leq \| \nabla (d_t + u \cdot \nabla d - |\nabla d|^2 d) \|_1^2 \\
&\leq \| \nabla d_t \|_1^2 + \| \nabla u \cdot \nabla d \|_1^2 + \| u \cdot \nabla^2 d \|_1^2 + \| \nabla (|\nabla d|^2 d) \|_1^2.
\end{aligned} \tag{3.179}$$

It is easy to check that

$$\| \nabla d_t \|_1^2 \leq \| \nabla d \|_{\mathcal{H}^2}^2 \leq \Lambda_m(t), \text{ for } m \geq 2, \tag{3.180}$$

$$\| \nabla u \cdot \nabla d \|_1^2 \leq \| (\nabla u, \nabla d) \|_{L^\infty}^2 \| (\nabla u, \nabla d) \|_1^2 \leq C_3 \Lambda_m^2(t), \text{ for } m \geq 2, \tag{3.181}$$

and

$$\| u \cdot \nabla^2 d \|_1^2 \leq \| u \|_{L^\infty}^2 \| \nabla^2 d \|^2 + \| Zu \|_{L^\infty}^2 \| \nabla^2 d \|^2 + \| u \|_{L^\infty}^2 \| \nabla^2 d \|_1^2 \leq C \Lambda_m^2(t), \text{ for } m \geq 3. \tag{3.182}$$

In view of the basic fact $|d| = 1$, one arrives at

$$\| \nabla (|\nabla d|^2 d) \|_1^2 \leq C \Lambda_m^3(t), \text{ } m \geq 3. \tag{3.183}$$

Substituting (3.180)-(3.183) into (3.179), we find

$$\|\nabla \Delta d\|_1^2 \leq C_3 \Lambda_m^3(t), \quad m \geq 3,$$

which, together with (3.178), yields immediately

$$\|\nabla^2 d\|_{L^\infty}^2 \leq C_4 P(\Lambda_m(t)), \quad m \geq 3. \quad (3.184)$$

Similarly, it is easy to check that for $|\alpha| = 1$

$$\|\mathcal{Z}^\alpha \nabla^2 d\|_{L^\infty}^2 \leq C_3 P(\Lambda_m(t)). \quad (3.185)$$

By virtue of the (1.1)₃, (3.174) and (3.184), one attains for $m \geq 3$

$$\begin{aligned} \|\nabla \Delta d\|_{L^\infty}^2 &\leq \|\nabla(d_t + u \cdot \nabla d - |\nabla d|^2 d)\|_{L^\infty}^2 \\ &\leq \|\nabla d_t\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2 \|\nabla d\|_{L^\infty}^2 + \|u\|_{L^\infty}^2 \|\nabla^2 d\|_{L^\infty}^2 \\ &\quad + \|\nabla d\|_{L^\infty}^2 \|\nabla^2 d\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^4 \|\nabla d\|_{L^\infty}^2 \\ &\leq C_4 \Lambda_m^3(t). \end{aligned} \quad (3.186)$$

Similarly, it is easy to check that for $|\alpha| = 1$

$$\|\mathcal{Z}^\alpha \nabla \Delta d\|_{L^\infty}^2 \leq C_{m+2} P(\Lambda_m(t)), \quad m \geq 3. \quad (3.187)$$

The combination of (3.170), (3.175) and (3.184)-(3.187) yields the estimate (3.166). Therefore, we complete the proof of Lemma 3.15. \square

In order to give the estimate for $\|\nabla u\|_{\mathcal{H}^{1,\infty}}$, we need the lemma as follows, refer to [12].

Lemma 3.16. *Let h be a smooth solution to*

$$a(t, y)[\partial_t h + b_1(t, y)\partial_{y_1} h + b_2(t, y)\partial_{y_2} h + z b_3(t, y)\partial_z h] - \varepsilon \partial_{zz} h = G, \quad z > 0, \quad h(t, y, 0) = 0, \quad (3.188)$$

for some smooth function $d(t, y) = \frac{1}{a(t, y)}$ and vector field $b = (b_1, b_2, b_3)^{tr}(t, y)$ satisfying (3.188). Assume that h and G are compactly supported in z . Then, it holds that

$$\begin{aligned} \|h\|_{\mathcal{H}^{1,\infty}} &\leq C \|h_0\|_{\mathcal{H}^{1,\infty}} + C \int_0^t \left\| \frac{1}{a} \right\|_{L^\infty} \|G\|_{\mathcal{H}^{1,\infty}} d\tau \\ &\quad + C \int_0^t \left(1 + \left\| \frac{1}{a} \right\|_{L^\infty} \right) \left(1 + \|b\|_{L^\infty}^2 + \sum_{i=0}^2 \|Z_i(a, b)\|_{L^\infty}^2 \right) \|h\|_{\mathcal{H}^{1,\infty}} d\tau. \end{aligned} \quad (3.189)$$

Finally, one gives the estimate for the quantity $\|\nabla u\|_{\mathcal{H}^{1,\infty}}$.

Lemma 3.17. *For $m \geq 6$, we have the estimate*

$$\begin{aligned} \|\nabla u\|_{\mathcal{H}^{1,\infty}}^2 &\leq C C_{m+2} \left\{ \|(u_0, \nabla u_0)\|_{\mathcal{H}^{1,\infty}}^2 + P(\Lambda_{1m}(t)) + P(\|\Delta p(t)\|_{\mathcal{H}^1}^2) + \varepsilon^2 t \int_0^t \|\nabla^2 u\|_{\mathcal{H}^4}^2 d\tau \right\} \\ &\quad + C C_{m+2} t \int_0^t (1 + P(\Lambda_m(\tau)) + P(Q(\tau)))(1 + \varepsilon^2 \|\Delta p\|_{\mathcal{H}^2}^2) d\tau. \end{aligned} \quad (3.190)$$

Proof. Away from the boundary, we clearly have by the classical isotropic Sobolev embedding that

$$\|\chi \nabla u\|_{L^\infty}^2 + \|\chi \mathcal{Z}^\alpha \nabla u\|_{L^\infty}^2 \lesssim \|u\|_{\mathcal{H}^m}^2, \quad m \geq 4, \quad |\alpha| = 1, \quad (3.191)$$

where the support of χ is away from the boundary. Consequently, by using a partition of unity subordinated to the covering we only have to estimate $\|\chi_j \nabla u\|_{L^\infty} + \|\chi_j \mathcal{Z}^\alpha \nabla u\|_{L^\infty}$, $j \geq 1$, $|\alpha| = 1$. For notational convenience, we shall denote χ_j by χ . Similar to [19] or [12], we use the local parametrization in the neighborhood of the boundary given by a normal geodesic system in which the Laplacian takes a convenient form. Denote

$$\Psi^n(y, z) = \begin{pmatrix} y \\ \psi(y) \end{pmatrix} - z n(y) = x,$$

where

$$n(y) = \frac{1}{\sqrt{1 + |\nabla\psi(y)|^2}} \begin{pmatrix} \partial_1\psi(y) \\ \partial_2\psi(y) \\ -1 \end{pmatrix}$$

is the unit outward normal. As before, one can extend n and Π in the interior by setting

$$n(\Psi^n(y, z)) = n(y), \quad \Pi(\Psi^n(y, z)) = \Pi(y) = I - n \otimes n,$$

where I is the unit matrix. Note that $n(y, z)$ and $\Pi(y, z)$ have different definitions from the ones used before. The advantages of this parametrization is that in the associated local basis (e_{y_1}, e_{y_2}, e_z) of \mathbb{R}^3 , it holds that $\partial_z = \partial_n$ and

$$(e_{y_i})|_{\Psi^n(y, z)} \cdot (e_z)|_{\Psi^n(y, z)} = 0, \quad i = 1, 2.$$

The scalar product on \mathbb{R}^3 induces in this coordinate system the Riemannian metric g with the norm

$$g(y, z) = \begin{pmatrix} \tilde{g}(y, z) & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the Laplacian in this coordinate system has the form

$$\Delta f = \partial_{zz}f + \frac{1}{2}\partial_z(\ln|g|)\partial_zf + \Delta_{\tilde{g}}f, \quad (3.192)$$

where $|g|$ denotes the determinant of the matrix g , and $\Delta_{\tilde{g}}$ is defined by

$$\Delta_{\tilde{g}}f = \frac{1}{\sqrt{|\tilde{g}|}} \sum_{i,j=1,2} \partial_{y_i}(\tilde{g}^{ij}|\tilde{g}|^{\frac{1}{2}}\partial_{y_j}f),$$

which only involves the tangential derivatives and $\{\tilde{g}^{ij}\}$ is the inverse matrix to g .

Next, thanks to (3.76) (in the coordinate system that we have just defined) and Lemma 3.15, we have for $m \geq 5$, $|\alpha| = 1$

$$\begin{aligned} & \|\chi \nabla u\|_{L^\infty}^2 + \|\chi \mathcal{Z}^\alpha \nabla u\|_{L^\infty}^2 \\ & \leq C_2(\|\chi \Pi(\partial_n u)\|_{L^\infty}^2 + \|\chi \operatorname{div} u\|_{L^\infty}^2 + \|\chi Z_y u\|_{L^\infty}^2) \\ & \quad + C_2(\|\chi \mathcal{Z}^\alpha \Pi(\partial_n u)\|_{L^\infty}^2 + \|\chi \mathcal{Z}^\alpha \operatorname{div} u\|_{L^\infty}^2 + \|\mathcal{Z}^\alpha(\chi Z_y u)\|_{L^\infty}^2) \\ & \leq C_3 \{ \|\chi \Pi \partial_n u\|_{L^\infty}^2 + \|\mathcal{Z}^\alpha(\chi \Pi \partial_n u)\|_{L^\infty}^2 + P(\Lambda_{1m}) + P(\|\Delta p\|_{\mathcal{H}^1}^2) \}. \end{aligned} \quad (3.193)$$

Consequently, it suffices to estimate $\|\chi \Pi \partial_n u\|_{\mathcal{H}^{1,\infty}}$. To this end, it is useful to use the vorticity $w = \nabla \times u$, see [11, 12, 19]. Indeed, it is easy to deduce that

$$\Pi(w \times n) = \Pi((\nabla u - \nabla u^t) \cdot n) = \Pi(\partial_n u - \nabla(u \cdot n) + \nabla n^t \cdot u),$$

which implies

$$\|\chi \Pi \partial_n u\|_{\mathcal{H}^{1,\infty}}^2 \leq C_3(\|\chi \Pi(w \times n)\|_{\mathcal{H}^{1,\infty}}^2 + P(\Lambda_{1m}(t))), \quad (3.194)$$

where we have used the Lemma 3.15. In other words, we only need to estimate $\|\chi \Pi(w \times n)\|_{\mathcal{H}^{1,\infty}}$. It is easy to see that w solves the vorticity equation

$$\rho w_t + \rho(u \cdot \nabla)w = \mu \varepsilon \Delta w + F_1, \quad (3.195)$$

where

$$F_1 \triangleq -\nabla \rho \times u_t - \nabla \rho \times (u \cdot \nabla)u + \rho(w \cdot \nabla)u - \rho w \operatorname{div} u - \nabla \times (\nabla d \cdot \Delta d).$$

In the support of χ , let

$$\tilde{w}(y, z) = w(\Psi^n(y, z)), \quad (\tilde{\rho}, \tilde{u}, \tilde{d})(y, z) = (\rho, u, d)(\Psi^n(y, z)),$$

The combination of (3.82) and (3.192) yields directly

$$\tilde{\rho} \partial_t \tilde{w} + \tilde{\rho} \tilde{u}^1 \partial_{y_1} \tilde{w} + \tilde{\rho} \tilde{u}^2 \partial_{y_2} \tilde{w} + \tilde{\rho} \tilde{u} \cdot n \partial_z \tilde{w} = \mu \varepsilon (\partial_{zz} \tilde{w} + \frac{1}{2} \partial_z(\ln|g|) \partial_z \tilde{w} + \Delta_{\tilde{g}} \tilde{w}) + \tilde{F}_1 \quad (3.196)$$

and

$$\tilde{\rho} \partial_t \tilde{u} + \tilde{\rho} \tilde{u}^1 \partial_{y_1} \tilde{u} + \tilde{\rho} \tilde{u}^2 \partial_{y_2} \tilde{u} + \tilde{\rho} \tilde{u} \cdot n \partial_z \tilde{u} = \mu \varepsilon (\partial_{zz} \tilde{u} + \frac{1}{2} \partial_z(\ln|g|) \partial_z \tilde{u} + \Delta_{\tilde{g}} \tilde{u}) + \tilde{F}_2, \quad (3.197)$$

where $\tilde{F}_2 = F_2(\Psi^n(y, z))$ and $F_2 = (\mu + \lambda)\varepsilon \nabla \operatorname{div} u - \nabla p - \nabla d \cdot \Delta d$. Similar to (3.78), we define

$$\tilde{\eta} = \chi(\tilde{w} \times n + \Pi(B\tilde{u})).$$

It is easy to deduce that $\tilde{\eta}$ satisfies

$$\tilde{\eta}(y, 0) = 0.$$

and solves the equation

$$\begin{aligned} & \tilde{\rho} \partial_t \tilde{\eta} + \tilde{\rho} \tilde{u}^1 \partial_{y_1} \tilde{\eta} + \tilde{\rho} \tilde{u}^2 \partial_{y_2} \tilde{\eta} + \tilde{\rho} \tilde{u} \cdot n \partial_z \tilde{\eta} \\ &= \mu \varepsilon \left(\partial_{zz} \tilde{\eta} + \frac{1}{2} \partial_z (\ln |g|) \partial_z \tilde{\eta} \right) + \chi(\tilde{F}_1 \times n) + \chi \Pi(B\tilde{F}_2) + F^\chi + \chi F^\kappa, \end{aligned} \quad (3.198)$$

where the source terms are given by

$$\begin{aligned} F^\chi &= [(\tilde{\rho} \tilde{u}^1 \partial_{y_1} + \tilde{\rho} \tilde{u}^2 \partial_{y_2} + \tilde{\rho} \tilde{u} \cdot n \partial_z) \chi] (\tilde{w} \times n + \Pi(B\tilde{u})) \\ &\quad - \mu \varepsilon \left(\partial_{zz} \chi + 2 \partial_z \chi \partial_z + \frac{1}{2} \partial_z (\ln |g|) \partial_z \chi \right) (\tilde{w} \times n + \Pi(B\tilde{u})), \end{aligned} \quad (3.199)$$

and

$$\begin{aligned} F^\kappa &= (\tilde{\rho} \tilde{u}^1 \partial_{y_1} \Pi + \tilde{\rho} \tilde{u}^2 \partial_{y_2} \Pi) \cdot (B\tilde{u}) + w \times (\tilde{\rho} \tilde{u}^1 \partial_{y_1} n + \tilde{\rho} \tilde{u}^2 \partial_{y_2} n) \\ &\quad + \Pi [(\tilde{\rho} \tilde{u}^1 \partial_{y_1} + \tilde{\rho} \tilde{u}^2 \partial_{y_2} + \tilde{\rho} \tilde{u} \cdot n \partial_z) B \cdot \tilde{u}] + \mu \varepsilon \Delta_{\tilde{g}} \tilde{w} \times n + \mu \varepsilon \Pi(B \Delta_{\tilde{g}} \tilde{u}). \end{aligned} \quad (3.200)$$

Note that in the derivation of the source terms above, in particular, F^κ , which contains all the commutators coming from the fact that n and Π are not constant, we have used the fact that in the coordinate system just defined, n and Π do not depend on the normal variable. Since $\Delta_{\tilde{g}}$ involves only the tangential derivatives, and the derivatives of χ are compactly supported away from the boundary, the following estimates hold for $m \geq 6$

$$\|F^\chi\|_{\mathcal{H}^{1,\infty}}^2 \leq C_3 (\|\rho u\|_{\mathcal{H}^{1,\infty}}^2 \|u\|_{\mathcal{H}^{2,\infty}}^2 + \varepsilon^2 \|u\|_{\mathcal{H}^{3,\infty}}^2) \leq C_3 \{P(Q(t)) + P(\Lambda_{1m})\}, \quad (3.201)$$

$$\|\chi(\tilde{F}_1 \times n)\|_{\mathcal{H}^{1,\infty}}^2 \leq C_2 (P(Q(t)) + \|\nabla d\|_{\mathcal{H}^{1,\infty}}^2 \|\nabla \Delta d\|_{\mathcal{H}^{1,\infty}}^2) \leq C_2 (P(Q(t)) + P(\Lambda_m)), \quad (3.202)$$

$$\begin{aligned} \|\chi \Pi(B\tilde{F}_2)\|_{\mathcal{H}^{1,\infty}}^2 &\leq C_3 (\varepsilon^2 \|\nabla \operatorname{div} u\|_{\mathcal{H}^{1,\infty}}^2 + \|\nabla p\|_{\mathcal{H}^{1,\infty}}^2 + \|\nabla d\|_{\mathcal{H}^{1,\infty}}^2 \|\Delta d\|_{\mathcal{H}^{1,\infty}}^2) \\ &\leq C_4 \{P(Q(t)) + P(\Lambda_m) + C\varepsilon^2 [1 + P(Q(t))] \|\Delta p\|_{\mathcal{H}^2}^2\}, \end{aligned} \quad (3.203)$$

and

$$\begin{aligned} \|\chi F^\kappa\|_{\mathcal{H}^{1,\infty}}^2 &\leq C_4 \{ \|u\|_{\mathcal{H}^{1,\infty}}^8 + \|u\|_{\mathcal{H}^{1,\infty}}^4 \|\nabla u\|_{\mathcal{H}^{1,\infty}}^4 + \|\rho\|_{\mathcal{H}^{1,\infty}}^2 \\ &\quad + \varepsilon^2 (\|\nabla u\|_{\mathcal{H}^{3,\infty}}^2 + \|u\|_{\mathcal{H}^{3,\infty}}^2) \} \\ &\leq C_4 \{ \varepsilon^2 \|\nabla^2 u\|_{\mathcal{H}^4}^2 + P(\Lambda_{1m}) + P(Q(t)) \}. \end{aligned} \quad (3.204)$$

It follows from (3.201)-(3.204) that

$$\|F\|_{\mathcal{H}^{1,\infty}}^2 \leq C_4 \{ \varepsilon^2 \|\nabla^2 u\|_{\mathcal{H}^4}^2 + P(Q(t)) + P(\Lambda_m) + \varepsilon^2 [1 + P(Q(t))] \|\Delta p\|_{\mathcal{H}^2}^2 \}, \quad (3.205)$$

where $\tilde{F} = \chi(\tilde{F}_1 \times n) + \chi \Pi(B\tilde{F}_2) + F^\chi + \chi F^\kappa$. In order to be able to use Lemma 3.16, we shall perform last change of unknown in order to eliminate the term $\partial_z (\ln |\tilde{g}|) \partial_z \tilde{\eta}$. We set

$$\tilde{\eta} = \frac{1}{|g|^{\frac{1}{4}}} \bar{\eta} = \bar{\gamma} \bar{\eta}.$$

Note that we have

$$\|\tilde{\eta}\|_{\mathcal{H}^{1,\infty}} \leq C_3 \|\bar{\eta}\|_{\mathcal{H}^{1,\infty}}, \quad \|\bar{\eta}\|_{\mathcal{H}^{1,\infty}} \leq C_3 \|\tilde{\eta}\|_{\mathcal{H}^{1,\infty}} \quad (3.206)$$

and that, $\bar{\eta}$ solves the equation

$$\begin{aligned} & \tilde{\rho} \partial_t \bar{\eta} + \tilde{\rho} \tilde{u}^1 \partial_{y_1} \bar{\eta} + \tilde{\rho} \tilde{u}^2 \partial_{y_2} \bar{\eta} + \tilde{\rho} \tilde{u} \cdot n \partial_z \bar{\eta} - \mu \varepsilon \partial_{zz} \bar{\eta} \\ &= \frac{1}{\bar{\gamma}} \left(\tilde{F} + \mu \varepsilon \partial_{zz} \bar{\gamma} \cdot \bar{\eta} + \frac{\mu \varepsilon}{2} \partial_z (\ln |g|) \partial_z \bar{\gamma} \cdot \bar{\eta} - \tilde{\rho} (\tilde{u} \cdot \nabla \bar{\gamma}) \bar{\eta} \right) := \mathcal{S}_1. \end{aligned}$$

In the spirit of Wang et al. [12], we rewrite the equation as follows

$$\tilde{\rho}(t, y, 0) [\bar{\eta}_t + \tilde{u}^1(t, y, 0) \partial_{y_1} + \tilde{u}^2(t, y, 0) \partial_{y_2} \bar{\eta} + z \partial_z (\tilde{u} \cdot n)(t, y, 0) \partial_z \bar{\eta}] - \mu \varepsilon \partial_{zz} \bar{\eta} = \mathcal{S}_1 + \mathcal{S}_2, \quad (3.207)$$

where \mathcal{S}_2 is defined as

$$\begin{aligned}\mathcal{S}_2 \triangleq & [\tilde{\rho}(t, y, 0) - \tilde{\rho}(t, y, z)]\eta_t + \sum_{i=1,2} [(\tilde{\rho} \tilde{u}^i)(t, y, 0) - (\tilde{\rho} \tilde{u}^i)(t, y, z)]\partial_{y_i}\bar{\eta} \\ & - \tilde{\rho}(t, y, z)[(\tilde{u} \cdot n)(t, y, z) - z\partial_z(\tilde{u} \cdot n)(t, y, 0)]\partial_z\bar{\eta} \\ & - [\tilde{\rho}(t, y, z) - \tilde{\rho}(t, y, 0)]z\partial_z(\tilde{u} \cdot n)(t, y, 0)\partial_z\bar{\eta}.\end{aligned}$$

Consequently, by using Lemma 3.16, we get from (3.207) that for $m \geq 6$

$$\begin{aligned}\|\bar{\eta}\|_{\mathcal{H}^{1,\infty}} & \lesssim C\|\bar{\eta}_0\|_{\mathcal{H}^{1,\infty}} + C \int_0^t \|\tilde{\rho}^{-1}\|_{L^\infty} \|(\mathcal{S}_1 + \mathcal{S}_2)\|_{\mathcal{H}^{1,\infty}} d\tau \\ & + C \int_0^t (1 + \|\tilde{\rho}^{-1}\|_{L^\infty})(1 + \|(\rho, u, \nabla u)\|_{\mathcal{H}^{1,\infty}}^2) \|\bar{\eta}\|_{\mathcal{H}^{1,\infty}} d\tau \\ & \lesssim C\|\bar{\eta}_0\|_{\mathcal{H}^{1,\infty}} + C \int_0^t \|(\mathcal{S}_1 + \mathcal{S}_2)\|_{\mathcal{H}^{1,\infty}} d\tau \\ & + C \int_0^t (1 + P(\Lambda_{1m}) + \|\mathcal{Z}\nabla u\|_{L^\infty}^2) \|\bar{\eta}\|_{\mathcal{H}^{1,\infty}} d\tau.\end{aligned}\tag{3.208}$$

On the other hand, following the same argument as [12], we have the following estimate

$$\|(\mathcal{S}_1 + \mathcal{S}_2)\|_{\mathcal{H}^{1,\infty}}^2 \leq C_4 \{ \varepsilon^2 \|\nabla^2 u\|_{\mathcal{H}^4}^2 + \varepsilon^2 [1 + P(Q(t))] \|\Delta p\|_{\mathcal{H}^2}^2 + P(Q(t)) + P(\Lambda_m) \}. \tag{3.209}$$

Then, we deduce from (3.208)-(3.209) that

$$\begin{aligned}\|\bar{\eta}(t)\|_{\mathcal{H}^{1,\infty}}^2 & \leq \|\bar{\eta}_0\|_{\mathcal{H}^{1,\infty}}^2 + C_4 t \int_0^t (1 + P(Q(\tau)) + P(\Lambda_m)) d\tau \\ & + C_4 t \varepsilon^2 \int_0^t ([1 + P(Q(t))] \|\Delta p\|_{\mathcal{H}^2}^2 + \|\nabla^2 u\|_{\mathcal{H}^4}^2) d\tau.\end{aligned}$$

which, together with (3.191), (3.206), (3.193), (3.194), completes the proof of Lemma 3.17. \square

3.5. Uniform estimate for Δp . In this subsection, we shall estimate Δp to complete the L^∞ -estimates and prove that the boundary layers for the density is weaker than the one for the velocity. Taking divergence operator to the equation (3.5), it is easy to deduce that

$$-\varepsilon \Delta \operatorname{div} u + \frac{1}{2\mu + \lambda} \Delta p = -\frac{1}{2\mu + \lambda} \operatorname{div}(\rho u_t + \rho u \cdot \nabla u) - \frac{1}{2\mu + \lambda} \operatorname{div}(\nabla d \cdot \Delta d). \tag{3.210}$$

On the other hand, it follows from the equation (1.1)₁ that

$$\operatorname{div} u = -(\ln \rho)_t - u \cdot \nabla \ln \rho = -\frac{p_t}{\gamma p} - \frac{u \cdot \nabla p}{\gamma p}. \tag{3.211}$$

Then the combination of (3.210) and (3.211) yields that

$$\begin{aligned}\varepsilon \Delta (\ln \rho)_t + \varepsilon u \cdot \nabla \Delta \ln \rho + \varepsilon \Delta u \cdot \nabla \ln \rho + 2\varepsilon \sum_{k=1}^3 \partial_k u \cdot \nabla \partial_k \ln \rho + \frac{1}{2\mu + \lambda} \Delta p \\ = -\frac{1}{2\mu + \lambda} \operatorname{div}(\rho u_t + \rho u \cdot \nabla u) - \frac{1}{2\mu + \lambda} \operatorname{div}(\nabla d \cdot \Delta d).\end{aligned}\tag{3.212}$$

Lemma 3.18. *For $m \geq 6$, it holds that*

$$\begin{aligned}\sup_{0 \leq \tau \leq t} (\|\Delta p(\tau)\|_{\mathcal{H}^1}^2 + \varepsilon \|\Delta p(\tau)\|_{\mathcal{H}^2}^2) + \int_0^t \|\Delta p(\tau)\|_{\mathcal{H}^2}^2 d\tau \\ \leq C C_{m+2} \left\{ P(N_m(0)) + [1 + P(Q(t))] \int_0^t P(N_m(\tau)) d\tau \right\}.\end{aligned}\tag{3.213}$$

Proof. Applying $\mathcal{Z}^\alpha (|\alpha| \leq 2)$ to (3.212) and multiplying by $\mathcal{Z}^\alpha \Delta \ln \rho$, it is easy to deduce that

$$\begin{aligned}
& \varepsilon \|\mathcal{Z}^\alpha \Delta \ln \rho\|^2 - \varepsilon \|\mathcal{Z}^\alpha \Delta \ln \rho_0\|^2 + \underbrace{\frac{2}{2\mu + \lambda} \int_0^t \int \mathcal{Z}^\alpha \Delta p \cdot \mathcal{Z}^\alpha \Delta \ln \rho dx d\tau}_{VII_1} \\
&= \underbrace{-2\varepsilon \int_0^t \int \mathcal{Z}^\alpha (\Delta u \cdot \nabla \ln \rho) \mathcal{Z}^\alpha \Delta \ln \rho dx d\tau}_{VII_2} - \underbrace{4\varepsilon \sum_{k=1}^3 \int_0^t \int \mathcal{Z}^\alpha (\partial_k u \cdot \nabla \partial_k \ln \rho) \mathcal{Z}^\alpha \Delta \ln \rho dx d\tau}_{VII_3} \\
&\quad - \underbrace{2\varepsilon \int_0^t \int \mathcal{Z}^\alpha (u \cdot \nabla \Delta \ln \rho) \mathcal{Z}^\alpha \Delta \ln \rho dx d\tau}_{VII_4} + \underbrace{\frac{-2}{2\mu + \lambda} \int_0^t \int \mathcal{Z}^\alpha \operatorname{div}(\rho u_t + \rho u \cdot \nabla u) \mathcal{Z}^\alpha \Delta \ln \rho dx d\tau}_{VII_5} \\
&\quad - \underbrace{\frac{2}{2\mu + \lambda} \int_0^t \int \mathcal{Z}^\alpha \operatorname{div}(\nabla d \cdot \Delta d) \mathcal{Z}^\alpha \Delta \ln \rho dx d\tau}_{VII_6}.
\end{aligned} \tag{3.214}$$

Using the same argument as Lemma 3.16 of [12], one can obtain the following estimates

$$VII_5 \leq C_{m+2} [1 + P(Q(t))] \int_0^t P(\Lambda_m) \tau, \tag{3.215}$$

$$VII_2 \leq \delta \int_0^t \|\mathcal{Z}^\alpha \Delta \ln \rho\|_{L^2}^2 d\tau + C_\delta [1 + P(Q(t))] \int_0^t P(\Lambda_m) d\tau, \tag{3.216}$$

$$\begin{aligned}
VII_3 &\leq \delta \int_0^t \|\mathcal{Z}^\alpha \Delta \ln \rho\|_{L^2}^2 d\tau + C_\delta \varepsilon^2 [1 + P(Q(t))] \int_0^t (P(\Lambda_m) + \|\Delta \ln \rho\|_{\mathcal{H}^2}^2) d\tau \\
&\quad + \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^3}^2 d\tau + C_\delta \varepsilon^2 \int_0^t \|\nabla u\|_{\mathcal{H}^4}^2 (\|\Delta \ln \rho\|_{L^2}^4 + P(\Lambda_m)) d\tau,
\end{aligned} \tag{3.217}$$

$$VII_1 \geq \frac{\gamma}{2} p(c_0) \int_0^t \|\mathcal{Z}^\alpha \Delta \ln \rho\|_{L^2}^2 d\tau - C [1 + P(Q(t))] \int_0^t (P(\Lambda_m) + \|\Delta \ln \rho\|_{\mathcal{H}^1}^2) d\tau, \tag{3.218}$$

$$VII_4 \leq \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^3}^2 d\tau + \delta \int_0^t \|\mathcal{Z}^\alpha \Delta \ln \rho\|_{L^2}^2 d\tau + C_\delta C_2 [1 + P(Q(t))] \int_0^t (\varepsilon^2 \|\Delta \ln \rho\|_{\mathcal{H}^2}^2 + P(\Lambda_m)) d\tau. \tag{3.219}$$

On the other hand, by using the Proposition 2.2, it is easy to check that

$$\begin{aligned}
VII_6 &\leq \delta \int_0^t \|\Delta \ln \rho\|_{\mathcal{H}^2}^2 d\tau + C_\delta \|\Delta d\|_{L^\infty}^2 \int_0^t \|\Delta d\|_{\mathcal{H}^2}^2 d\tau \\
&\quad + C_\delta \|\nabla d\|_{L^\infty}^2 \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^2}^2 d\tau + C_\delta \|\nabla \Delta d\|_{L^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^2}^2 d\tau \\
&\leq \delta \int_0^t \|\Delta \ln \rho\|_{\mathcal{H}^2}^2 d\tau + C_\delta [1 + P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau.
\end{aligned} \tag{3.220}$$

Hence, the combination of (3.215)-(3.220) gives directly

$$\begin{aligned}
& \varepsilon \|\Delta \ln \rho\|_{\mathcal{H}^2}^2 + \int_0^t \|\Delta \ln \rho\|_{\mathcal{H}^2}^2 d\tau \\
&\leq C\varepsilon \|\Delta \ln \rho_0\|_{\mathcal{H}^2}^2 + C\varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^3}^2 d\tau \\
&\quad + C_\delta C_{m+2} [1 + P(Q(t))] \int_0^t (\varepsilon \|\Delta \ln \rho\|_{\mathcal{H}^2}^2 + \|\Delta \ln \rho\|_{\mathcal{H}^1}^4 + P(\Lambda_m)) d\tau.
\end{aligned} \tag{3.221}$$

On the other hand, it is easy to obtain that

$$\|\Delta \ln \rho(t)\|_{\mathcal{H}^1}^2 \leq \|\Delta \ln \rho_0\|_{\mathcal{H}^1}^2 + \int_0^t \|\Delta \ln \rho(\tau)\|_{\mathcal{H}^2}^2 d\tau,$$

which, together with (3.221), yields directly

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} (\|\Delta \ln \rho(\tau)\|_{\mathcal{H}^1}^2 + \varepsilon \|\Delta \ln \rho(\tau)\|_{\mathcal{H}^2}^2) + \int_0^t \|\Delta \ln \rho(\tau)\|_{\mathcal{H}^2}^2 d\tau \\ & \leq CC_{m+2} \left\{ N_m(0) + [1 + P(Q(t))] \int_0^t (1 + \varepsilon \|\Delta \ln \rho(\tau)\|_{\mathcal{H}^2}^2 + \|\Delta \ln \rho(\tau)\|_{\mathcal{H}^1}^4 + P(\Lambda_m)) d\tau \right\}. \end{aligned}$$

Therefore, we complete the proof of Lemma 3.18. \square

3.6. Proof of Theorem 3.1. By virtue of (3.161), (3.162) and (3.166), it is easy to deduce that

$$\begin{aligned} Q(t) & \leq C_3 \sup_{0 \leq \tau \leq t} \{ \|\nabla u(\tau)\|_{\mathcal{H}^{1,\infty}}^2 + P(\Lambda_m(\tau)) + P(\|\Delta p(\tau)\|_{\mathcal{H}^1}^2) \} \\ & \leq CC_{m+2} \left\{ P(N_m(0)) + P(N_m(t)) \int_0^t P(N_m(\tau)) d\tau \right\}. \end{aligned} \quad (3.222)$$

In order to close the a priori estimates, one still need to get the uniform estimate for $\|\nabla \partial_t^{m-1} u\|_{L^2}^2$. To this end, we combine (3.160), (3.222) and Lemma 3.11 to deduce that

$$\begin{aligned} \|\nabla \partial_t^{m-1} u\|_{L^2}^2 & \leq CC_{m+2} \{ \|u(t)\|_{\mathcal{H}^m}^2 + \|\eta(t)\|_{\mathcal{H}^{m-1}}^2 + \|\partial_t^{m-1} \operatorname{div} u\|_{L^2}^2 \} \\ & \leq CC_{m+2} \{ P(\Lambda_{1m}) + \|\eta(t)\|_{\mathcal{H}^{m-1}}^2 + P(Q(t)) \} \\ & \leq CC_{m+2} \left\{ P(N_m(0)) + P(N_m(t)) \int_0^t P(N_m(\tau)) d\tau \right\}. \end{aligned} \quad (3.223)$$

Hence, the combination of (3.160), (3.213), (3.222) and (3.223) yields for $m \geq 6$ that

$$\begin{aligned} & N_m(t) + \int_0^t (\|\nabla \partial_t^{m-1} p(\tau)\|_{L^2}^2 + \|\Delta p(\tau)\|_{\mathcal{H}^2}^2) d\tau + \varepsilon \int_0^t \|\nabla u(\tau)\|_{\mathcal{H}^m}^2 d\tau \\ & + \varepsilon \sum_{k=0}^{m-2} \int_0^t \|\nabla^2 \partial_t^k u(\tau)\|_{m-1-k}^2 d\tau + \varepsilon^2 \int_0^t \|\nabla^2 \partial_t^{m-1} u(\tau)\|_{L^2}^2 d\tau \\ & + \int_0^t \|\Delta d(\tau)\|_{\mathcal{H}^m}^2 d\tau + \int_0^t \|\nabla \Delta d(\tau)\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & \leq \tilde{C}_2 CC_{m+2} \left\{ P(N_m(0)) + P(N_m(t)) \int_0^t P(N_m(\tau)) d\tau \right\}, \quad \forall t \in [0, T], \end{aligned}$$

which completes the proof of (3.3). Furthermore, the (1.1)₃ implies that

$$|\rho(x, 0)| \exp \left(- \int_0^t \|\operatorname{div} u(\tau)\|_{L^\infty} d\tau \right) \leq \rho(x, t) \leq |\rho(x, 0)| \exp \left(\int_0^t \|\operatorname{div} u(\tau)\|_{L^\infty} d\tau \right),$$

which proves (3.1). Therefore, we complete the proof of Theorem 3.1.

4. PROOF OF THEOREM 1.1 (UNIFORM REGULARITY)

In this section, we will give the proof for the Theorem 1.1. Indeed, we shall indicate how to combine the a priori estimates obtained so far to prove the uniform existence result. Fixing $m \geq 6$, we consider the initial data $(p_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon) \in X_{n,m}^\varepsilon$ such that

$$\mathcal{I}_m(0) = \sup_{0 < \varepsilon \leq 1} \|(p_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon)\|_{X_{n,m}^\varepsilon} \leq \tilde{C}_0 \quad \text{and} \quad 0 < \widehat{C}_0^{-1} \leq \rho_0^\varepsilon \leq \widehat{C}_0. \quad (4.1)$$

For such initial data, since we are not aware of a local existence result for (1.1) and (1.2) (or (1.3)), we first establish the local existence of solution for (1.1) and (1.2) with initial data $(p_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon) \in X_{n,m}^\varepsilon$. For such initial data $(p_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon)$, it is easy to see that there exists a sequence of smooth approximate

initial data $(p_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}, d_0^{\varepsilon,\delta}) \in X_{n,m}^{\varepsilon,ap}$ (δ being a regularity parameter), which have enough space regularity so that the time derivatives at the initial time can be defined by the equations (1.1) and the boundary compatibility conditions are satisfied. Fixed $\varepsilon \in (0, 1]$, one constructs the approximate solutions as follows:

(1) Define $u^0 = u_0^{\varepsilon,\delta}$ and $d^0 = d_0^{\varepsilon,\delta}$.

(2) Assume that (u^{k-1}, d^{k-1}) has been defined for $k \geq 1$. Let (ρ^k, u^k, d^k) be the unique solution to the following linearized initial data boundary value problem:

$$\begin{cases} \rho_t^k + \operatorname{div}(\rho^k u^{k-1}) = 0, & \text{in } \Omega \times (0, T), \\ \rho^k u_t^k + \rho^k u^{k-1} \cdot \nabla u^k + \nabla p^k = \varepsilon \mu \Delta u^k + \varepsilon(\mu + \lambda) \nabla \operatorname{div} u^k - \nabla d^k \cdot \Delta d^k, & \text{in } \Omega \times (0, T), \\ d_t^k - \Delta d^k = |\nabla d^{k-1}|^2 d^{k-1} - u^{k-1} \cdot \nabla d^{k-1}, & \text{in } \Omega \times (0, T), \end{cases} \quad (4.2)$$

with initial data

$$(\rho^k, u^k, d^k)|_{t=0} = (\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}, d_0^{\varepsilon,\delta}), \quad (4.3)$$

and Navier-type and Neumann boundary condition

$$u^k \cdot n = 0, \quad n \times (\nabla \times u^k) = [Bu^k]_\tau, \quad \text{and} \quad \frac{\partial d^k}{\partial n} = 0, \quad \text{on } \partial\Omega. \quad (4.4)$$

Since ρ^k, u^k and d^k are decoupled, the existence of global unique smooth solution $(\rho^k, u^k, d^k)(t)$ of (4.2)-(4.4) can be obtained by using classical methods, for example, the similar argument in Cho et al. [27]. On the other hand, by virtue of $(\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}, d_0^{\varepsilon,\delta}) \in H^{4m} \times H^{4m} \times H^{4(m+1)}$, one proves that there exists a positive time $\tilde{T}_1 = \tilde{T}_1(\varepsilon)$ (depending on $\varepsilon, \|(\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta})\|_{H^{4m}}$ and $\|d_0^{\varepsilon,\delta}\|_{H^{4(m+1)}}$) such that

$$\|(\rho^k, u^k)(t)\|_{H^{4m}}^2 + \|d^k(t)\|_{H^{4(m+1)}}^2 \leq \tilde{C}_1 \quad \text{and} \quad \frac{\hat{C}_0}{2} \leq \rho^k(t) \leq 2\hat{C}_0 \quad \text{for } 0 \leq t \leq \tilde{T}_1, \quad (4.5)$$

where the constant \tilde{C}_1 depends on $\tilde{C}_0, \varepsilon^{-1}, \|(\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta})\|_{H^{4m}}$ and $\|d_0^{\varepsilon,\delta}\|_{H^{4(m+1)}}$. Based on the above uniform time $\hat{T}_1(\leq \tilde{T}_1)$ (independent of k) such that (ρ^k, u^k, d^k) converges to a limit $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, d^{\varepsilon,\delta})$ as $k \rightarrow +\infty$ in the following strong sense:

$$(\rho^k, u^k) \rightarrow (\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}) \text{ in } L^\infty(0, \hat{T}_1; L^2) \quad \text{and} \quad \nabla u^k \rightarrow \nabla u^{\varepsilon,\delta} \text{ in } L^2(0, \hat{T}_1; L^2),$$

and

$$d^k \rightarrow d^{\varepsilon,\delta} \text{ in } L^\infty(0, \hat{T}_1; H^1) \quad \text{and} \quad \Delta d^k \rightarrow \Delta d^{\varepsilon,\delta} \text{ in } L^2(0, \hat{T}_1; L^2).$$

It is easy to check that $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, d^{\varepsilon,\delta})$ is a classical solution to the problem (1.1) and (1.2) with initial data $(\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}, d_0^{\varepsilon,\delta})$. In view of the lower semicontinuity of norms, one can deduce from the uniform bounds (4.6) that

$$\|(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})(t)\|_{H^{4m}}^2 + \|d^{\varepsilon,\delta}(t)\|_{H^{4(m+1)}}^2 \leq \tilde{C}_1 \quad \text{and} \quad \frac{\hat{C}_0}{2} \leq \rho^k(t) \leq 2\hat{C}_0 \quad \text{for } 0 \leq t \leq \tilde{T}_1. \quad (4.6)$$

Applying the a priori estimates given in Theorem 3.1 to the solution $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, d^{\varepsilon,\delta})$, one can obtain a uniform time T_0 and constant C_3 (independent of ε and δ) such that

$$\begin{aligned} & N_m(p^{\varepsilon,\delta}, u^{\varepsilon,\delta}, d^{\varepsilon,\delta})(t) + \int_0^t (\|\nabla \partial_t^{m-1} p^{\varepsilon,\delta}(\tau)\|_{L^2}^2 + \|\Delta p^{\varepsilon,\delta}(\tau)\|_{\mathcal{H}^2}^2) d\tau \\ & + \varepsilon \int_0^t \|\nabla u^{\varepsilon,\delta}(\tau)\|_{\mathcal{H}^m}^2 d\tau + \varepsilon \sum_{k=0}^{m-2} \int_0^t \|\nabla^2 \partial_t^k u^{\varepsilon,\delta}(\tau)\|_{m-1-k}^2 d\tau \\ & + \varepsilon^2 \int_0^t \|\nabla^2 \partial_t^{m-1} u^{\varepsilon,\delta}(\tau)\|_{L^2}^2 d\tau + \int_0^t \|\Delta d^{\varepsilon,\delta}(\tau)\|_{\mathcal{H}^m}^2 d\tau + \int_0^t \|\nabla \Delta d^{\varepsilon,\delta}(\tau)\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & \leq \tilde{C}_3, \quad \forall t \in [0, \min\{T_0, \hat{T}_1\}], \end{aligned} \quad (4.7)$$

and

$$\frac{1}{2\widehat{C}_0} \leq \rho^{\varepsilon,\delta}(t) \leq 2\widehat{C}_0, \quad t \in [0, \min\{T_0, \widehat{T}_1\}]. \quad (4.8)$$

where T_0 and \widehat{C}_3 depend only on \widehat{C}_0 and $\mathcal{I}_m(0)$. Based on the uniform estimate (4.7) and (4.8) for $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, d^{\varepsilon,\delta})$, one can pass the limit $\delta \rightarrow 0$ to get a strong solution $(\rho^\varepsilon, u^\varepsilon, d^\varepsilon)$ of (1.1) and (1.2) with initial data $(\rho_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon)$ satisfying (4.2) by using a strong compactness arguments (see [28]). Indeed, it follows from (4.7) that $(p^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \nabla d^{\varepsilon,\delta})$ is bounded uniformly in $L^\infty(0, \widehat{T}_2; H_{co}^m)$, where $\widehat{T}_2 = \min\{T_0, \widehat{T}_1\}$, while $(\nabla p^{\varepsilon,\delta}, \nabla u^{\varepsilon,\delta}, \Delta d^{\varepsilon,\delta})$ is bounded uniformly in $L^\infty(0, \widehat{T}_2; H_{co}^{m-1})$, and $(\partial_t p^{\varepsilon,\delta}, \partial_t u^{\varepsilon,\delta}, \partial_t \nabla d^{\varepsilon,\delta})$ is bounded uniformly in $L^\infty(0, \widehat{T}_2; H_{co}^{m-1})$. Then, the strong compactness argument implies that $(p^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \nabla d^{\varepsilon,\delta})$ is compact in $\mathcal{C}([0, \widehat{T}_2]; H_{co}^{m-1})$. In particular, there exists a sequence $\delta_n \rightarrow 0^+$ and $(p^\varepsilon, u^\varepsilon, \nabla d^\varepsilon) \in \mathcal{C}([0, \widehat{T}_2]; H_{co}^{m-1})$ such that

$$(p^{\varepsilon,\delta_n}, u^{\varepsilon,\delta_n}, \nabla d^{\varepsilon,\delta_n}) \rightarrow (p^\varepsilon, u^\varepsilon, \nabla d^\varepsilon) \text{ in } \mathcal{C}([0, \widehat{T}_2]; H_{co}^{m-1}) \text{ as } \delta_n \rightarrow 0^+.$$

Moreover, applying the lower semicontinuity of norms to the bounds (4.7), one obtains the bounds (4.7) and (4.8) for $(p^\varepsilon, u^\varepsilon, d^\varepsilon)$. It follows from the bounds of (4.7) and (4.8) for $(p^\varepsilon, u^\varepsilon, d^\varepsilon)$, and the anisotropic Sobolev inequality (2.5) that

$$\begin{aligned} & \sup_{0 \leq t \leq \widehat{T}_2} \|(\rho^{\varepsilon,\delta_n} - \rho^\varepsilon, u^{\varepsilon,\delta_n} - u^\varepsilon, d^{\varepsilon,\delta_n} - d^\varepsilon)(t)\|_{L^\infty}^2 \\ & \leq C \sup_{0 \leq t \leq \widehat{T}_2} \|\nabla(\rho^{\varepsilon,\delta_n} - \rho^\varepsilon, u^{\varepsilon,\delta_n} - u^\varepsilon, d^{\varepsilon,\delta_n} - d^\varepsilon)\|_{H_{co}^1} \|(\rho^{\varepsilon,\delta_n} - \rho^\varepsilon, u^{\varepsilon,\delta_n} - u^\varepsilon, d^{\varepsilon,\delta_n} - d^\varepsilon)\|_{H_{co}^2} \rightarrow 0, \end{aligned}$$

and

$$\sup_{0 \leq t \leq \widehat{T}_2} \|\nabla(d^{\varepsilon,\delta_n} - d^\varepsilon)\|_{L^\infty}^2 \leq C \sup_{0 \leq t \leq \widehat{T}_2} \|\Delta(d^{\varepsilon,\delta_n} - d^\varepsilon)\|_{H_{co}^1} \|\nabla(d^{\varepsilon,\delta_n} - d^\varepsilon)\|_{H_{co}^2} \rightarrow 0,$$

Hence, it is easy to check that $(\rho^\varepsilon, u^\varepsilon, d^\varepsilon)$ is a weak solution of the nematic liquid crystal flows (1.1). The uniqueness of the solution $(\rho^\varepsilon, u^\varepsilon, d^\varepsilon)$ comes directly from the Lipschitz regularity of solution. Thus, the whole family $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, d^{\varepsilon,\delta})$ converge to $(\rho^\varepsilon, u^\varepsilon, d^\varepsilon)$. Therefore, we have established the local solution of equation (1.1) and (1.2) with initial data $(p_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon) \in X_{n,m}^\varepsilon$, $t \in [0, T_2]$.

We shall use the local existence results to prove Theorem 1.1. If $T_0 \leq \widehat{T}$, then Theorem 1.1 follows from (4.7) and (4.8) with $\widehat{C}_1 = \widehat{C}_3$. On the other hand, for the case $\widehat{T} \leq T_0$, based on the uniform estimate (4.7) and (4.8), we can use the local existence results established above to extend our solution step by step to the uniform time interval $t \in [0, T_0]$. Therefore, we complete the proof of Theorem 1.1.

5. PROOF OF THEOREM 1.3 (INVISCID LIMIT)

In this section, we study the vanishing viscosity of solutions for the equation (1.1) to the solution for the equation (1.4) with a rate of convergence. It is easy to see that the solution $(\rho, u, d) \in H^3 \times H^3 \times H^4$ of equation (1.1) and (1.2) with initial data $(\rho_0, u_0, d_0) \in H^3 \times H^3 \times H^4$ satisfies

$$\sum_{k=0}^3 \|(\rho, u)\|_{C^k([0, T_1]; H^{3-k})} + \sum_{k=0}^2 \|d\|_{C^k([0, T_1]; H^{4-2k})} \leq \widetilde{C}_4$$

where \widetilde{C}_4 depends only on $\|(\rho_0, u_0, d_0)\|_{H^3 \times H^3 \times H^4}$. On the other hand, it follows from the Theorem 1.1 that the solution $(\rho^\varepsilon, u^\varepsilon, d^\varepsilon)$ of equation (1.1) and (1.2) with initial data (ρ_0, u_0, d_0) satisfies

$$\|(p(\rho^\varepsilon), u^\varepsilon, d^\varepsilon)\|_{X_m^\varepsilon} \leq \widetilde{C}_1, \quad \frac{1}{2\widehat{C}_0} \leq \rho^\varepsilon(t) \leq 2\widehat{C}_0 \quad \forall t \in [0, T_0],$$

where T_0 and \widetilde{C}_1 are defined in Theorem 1.1. In particular, this uniform regularity implies the bound

$$\|(\rho^\varepsilon, u^\varepsilon)\|_{W^{1,\infty}} + \|d^\varepsilon\|_{W^{2,\infty}} + \|\partial_t(\rho^\varepsilon, u^\varepsilon)\|_{L^\infty} + \|d_t^\varepsilon\|_{W^{1,\infty}} \leq \widetilde{C}_1,$$

which plays an important role in the proof of Theorem 1.3.

Let us define

$$\phi^\varepsilon = \rho^\varepsilon - \rho, \quad v^\varepsilon = u^\varepsilon - u, \quad \varphi^\varepsilon = d^\varepsilon - d.$$

It then follows from (1.1) that

$$\begin{cases} \partial_t \phi^\varepsilon + \rho \operatorname{div} v^\varepsilon + u \cdot \nabla \phi^\varepsilon = R_1^\varepsilon, \\ \rho \partial_t v^\varepsilon + \rho u \cdot \nabla v^\varepsilon + \nabla(p^\varepsilon - p) + \Phi^\varepsilon = -\mu \varepsilon \nabla \times (\nabla \times v^\varepsilon) + (2\mu + \lambda) \varepsilon \nabla \operatorname{div} v^\varepsilon + R_2^\varepsilon + R_3^\varepsilon, \\ \partial_t \varphi^\varepsilon - \Delta \varphi^\varepsilon = R_4^\varepsilon, \end{cases} \quad (5.1)$$

where

$$\begin{aligned} R_1^\varepsilon &= -\phi^\varepsilon \operatorname{div} v^\varepsilon - v^\varepsilon \cdot \nabla \phi^\varepsilon - \phi^\varepsilon \operatorname{div} u - \nabla \rho \cdot v^\varepsilon, \\ R_2^\varepsilon &= -\phi^\varepsilon v_t^\varepsilon - \phi^\varepsilon u_t + \mu \varepsilon \Delta u + (\mu + \lambda) \varepsilon \nabla \operatorname{div} u, \\ R_3^\varepsilon &= -\nabla d^\varepsilon \cdot \Delta \varphi^\varepsilon - \nabla \varphi^\varepsilon \cdot \Delta d, \\ R_4^\varepsilon &= -u \cdot \nabla \varphi^\varepsilon - v^\varepsilon \cdot \nabla d^\varepsilon + (\nabla \varphi^\varepsilon : \nabla (d^\varepsilon + d)) d^\varepsilon + |\nabla d|^2 \varphi^\varepsilon, \\ \Phi^\varepsilon &= (\rho^\varepsilon u^\varepsilon - \rho u) \cdot \nabla u^\varepsilon. \end{aligned}$$

The boundary conditions to (5.1) are given as follows

$$\begin{cases} v^\varepsilon \cdot n = 0, & n \times (\nabla \times v^\varepsilon) = [Bv^\varepsilon]_\tau + [Bu]_\tau - n \times w, \quad x \in \partial\Omega, \\ \frac{\partial \varphi^\varepsilon}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \quad (5.2)$$

Lemma 5.1. *For $t \in [0, \min\{T_0, T_1\}]$, it holds that*

$$\sup_{0 \leq \tau \leq t} (\|(\phi^\varepsilon, v^\varepsilon)(\tau)\|_{L^2}^2 + \|\varphi^\varepsilon(\tau)\|_{H^1}^2) + \mu \varepsilon \int_0^t \|v^\varepsilon\|_{H^1}^2 d\tau + \int_0^t (\|\nabla \varphi^\varepsilon\|_{L^2}^2 + \|\Delta \varphi^\varepsilon\|_{L^2}^2) d\tau \leq C \varepsilon^{\frac{3}{2}}. \quad (5.3)$$

where $C > 0$ depend only on \tilde{C}_0, \tilde{C}_1 and \tilde{C}_4 .

Proof. Multiplying (5.1)₁ by v^ε , it is easy to deduce that

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \int \rho |v^\varepsilon|^2 dx + \int \Phi^\varepsilon \cdot v^\varepsilon dx + \int \nabla(p^\varepsilon - p) \cdot v^\varepsilon dx \\ &= -\mu \varepsilon \int \nabla \times (\nabla \times v^\varepsilon) \cdot v^\varepsilon dx + (2\mu + \lambda) \varepsilon \int \nabla \operatorname{div} v^\varepsilon \cdot v^\varepsilon dx + \int R_2^\varepsilon \cdot v^\varepsilon dx + \int R_3^\varepsilon \cdot v^\varepsilon dx. \end{aligned} \quad (5.4)$$

It is easy to check that

$$\int \Phi^\varepsilon \cdot v^\varepsilon dx \leq C \|(\rho, u^\varepsilon, \nabla u^\varepsilon)\|_{L^\infty} (\|\phi^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2). \quad (5.5)$$

Integrating by part and applying the equation (1.1)₁, we find

$$\begin{aligned} & \int \nabla(p^\varepsilon - p) \cdot v^\varepsilon dx = - \int (p^\varepsilon - p) \operatorname{div} v^\varepsilon dx \\ & \geq \int \frac{p'(\rho)}{\rho} \phi^\varepsilon (\phi_t^\varepsilon + u \cdot \nabla \phi^\varepsilon - R_1^\varepsilon) dx - C(1 + \|\nabla u^\varepsilon\|_{L^\infty}) \|\phi^\varepsilon\|_{L^2}^2 \\ & \geq \frac{d}{dt} \int \frac{p'(\rho)}{2\rho} |\phi^\varepsilon|^2 dx - C(1 + \|(\rho, u, \rho^\varepsilon, u^\varepsilon)\|_{W^{1,\infty}}) (\|\phi^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2) \\ & \geq \frac{d}{dt} \int \frac{p'(\rho)}{2\rho} |\phi^\varepsilon|^2 dx - C(\|\phi^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2). \end{aligned} \quad (5.6)$$

Integrating by part and applying the boundary condition (5.2), one arrives at directly

$$\begin{aligned}
& -\mu\varepsilon \int \nabla \times (\nabla \times v^\varepsilon) \cdot v^\varepsilon dx \\
& = -\mu\varepsilon \int_{\partial\Omega} n \times (\nabla \times v^\varepsilon) \cdot v^\varepsilon d\sigma - \mu\varepsilon \int |\nabla \times v^\varepsilon|^2 dx \\
& = -\mu\varepsilon \int_{\partial\Omega} ([Bv^\varepsilon]_\tau + [Bu]_\tau - n \times w) \cdot v^\varepsilon d\sigma - \mu\varepsilon \int |\nabla \times v^\varepsilon|^2 dx \\
& \leq -\mu\varepsilon \|\nabla \times v^\varepsilon\|_{L^2}^2 + C\varepsilon(|v^\varepsilon|_{L^2(\partial\Omega)}^2 + |v^\varepsilon|_{L^2(\partial\Omega)}),
\end{aligned} \tag{5.7}$$

and

$$(2\mu + \lambda)\varepsilon \int \nabla \operatorname{div} v^\varepsilon \cdot v^\varepsilon dx = (2\mu + \lambda)\varepsilon \int |\operatorname{div} v^\varepsilon|^2 dx. \tag{5.8}$$

On the other hand, by virtue of the Hölder and Young inequalities, one attains

$$\int R_2^\varepsilon \cdot v^\varepsilon dx \leq C\|(\phi^\varepsilon, v^\varepsilon)\|_{L^2}^2 + C\varepsilon^2, \tag{5.9}$$

and

$$\int R_3^\varepsilon \cdot v^\varepsilon dx \leq \delta \|\Delta \varphi^\varepsilon\|_{L^2}^2 + C_\delta(\|v^\varepsilon\|_{L^2}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2). \tag{5.10}$$

Substituting (5.5)-(5.10) into (5.4), we obtain

$$\begin{aligned}
& \frac{d}{dt} \int \left(\frac{p'(\rho)}{\rho} |\phi^\varepsilon|^2 + \frac{\rho}{2} |v^\varepsilon|^2 \right) dx + \mu\varepsilon \|\nabla \times v^\varepsilon\|_{L^2}^2 + (2\mu + \lambda)\varepsilon \|\operatorname{div} v^\varepsilon\|_{L^2}^2 \\
& \leq C_\delta \|(\phi^\varepsilon, v^\varepsilon, \nabla \varphi^\varepsilon)\|_{L^2}^2 + C\varepsilon(|v^\varepsilon|_{L^2(\partial\Omega)}^2 + |v^\varepsilon|_{L^2(\partial\Omega)}) + C\varepsilon^2 + \delta \|\Delta \varphi^\varepsilon\|_{L^2}^2.
\end{aligned} \tag{5.11}$$

The application of Proposition 2.1 gives directly

$$\|\nabla v^\varepsilon\|_{H^1}^2 \leq C(\|\nabla \times v^\varepsilon\|_{L^2}^2 + \|\operatorname{div} v^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2). \tag{5.12}$$

By virtue of the trace theorem in Proposition 2.3 and Cauchy inequality, one deduces that

$$|v^\varepsilon|_{L^2(\partial\Omega)}^2 \leq \delta \|\nabla v^\varepsilon\|_{L^2}^2 + C_\delta \|v^\varepsilon\|_{L^2}^2, \tag{5.13}$$

and

$$\begin{aligned}
\varepsilon |v^\varepsilon|_{L^2(\partial\Omega)} & \leq \varepsilon \|v^\varepsilon\|_{L^2}^{\frac{1}{2}} \|\nabla v^\varepsilon\|_{L^2}^{\frac{1}{2}} \leq \delta \varepsilon \|\nabla v^\varepsilon\|_{L^2}^2 + C_\delta \varepsilon \|v^\varepsilon\|_{L^2}^{\frac{3}{2}} \\
& \leq \delta \varepsilon \|\nabla v^\varepsilon\|_{L^2}^2 + C_\delta \|v^\varepsilon\|_{L^2}^2 + \varepsilon^{\frac{3}{2}}.
\end{aligned} \tag{5.14}$$

Then, the combination of (5.11)-(5.14) yields that

$$\frac{d}{dt} \int \left(\frac{p'(\rho)}{\rho} |\phi^\varepsilon|^2 + \frac{\rho}{2} |v^\varepsilon|^2 \right) dx + \mu\varepsilon \|v^\varepsilon\|_{H^1}^2 \leq C\|(\phi^\varepsilon, v^\varepsilon, \nabla \varphi^\varepsilon)\|_{L^2}^2 + C\varepsilon^{\frac{3}{2}} + \delta \|\Delta \varphi^\varepsilon\|_{L^2}^2. \tag{5.15}$$

Multiplying (5.1) by $-\Delta \varphi^\varepsilon$ and integrating over Ω , we find

$$-\int \partial_t \varphi^\varepsilon \cdot \Delta \varphi^\varepsilon dx + \int |\Delta \varphi^\varepsilon|^2 dx = -\int R_3^\varepsilon \cdot \Delta \varphi^\varepsilon dx. \tag{5.16}$$

Integrating by part and applying the boundary condition (5.2), it holds that

$$-\int \partial_t \varphi^\varepsilon \cdot \Delta \varphi^\varepsilon dx = -\int_{\partial\Omega} \partial_t \varphi^\varepsilon \cdot (n \cdot \nabla \varphi^\varepsilon) d\sigma + \frac{1}{2} \frac{d}{dt} \int |\nabla \varphi^\varepsilon|^2 dx = \frac{1}{2} \frac{d}{dt} \int |\nabla \varphi^\varepsilon|^2 dx. \tag{5.17}$$

Applying the Cauchy inequality, it is easy to deduce that

$$\begin{aligned}
-\int R_2^\varepsilon \cdot \Delta \varphi^\varepsilon dx & \leq \delta \|\Delta \varphi^\varepsilon\|_{L^2}^2 + C_\delta \|u\|_{L^\infty}^2 \|\nabla \varphi^\varepsilon\|_{L^2}^2 + C_\delta \|\nabla d^\varepsilon\|_{L^\infty}^2 (\|\nabla \varphi^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2) \\
& \quad + C_\delta \|\nabla d\|_{L^\infty}^2 (\|\nabla \varphi^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{L^2}^2) \\
& \leq \delta \|\Delta \varphi^\varepsilon\|_{L^2}^2 + C_\delta (\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{L^2}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2).
\end{aligned} \tag{5.18}$$

Substituting (5.17)-(5.18) into (5.16), we obtain

$$\frac{1}{2} \frac{d}{dt} \int |\nabla \varphi^\varepsilon|^2 dx + \frac{3}{4} \int |\Delta \varphi^\varepsilon|^2 dx \leq C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{L^2}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2). \quad (5.19)$$

In order to control the term $\int |\varphi^\varepsilon|^2 dx$ on the right hand side of (5.19), we multiply the equation (5.1)₃ by φ^ε and integrating by part to get that

$$\frac{1}{2} \frac{d}{dt} \int |\varphi^\varepsilon|^2 dx + \int |\nabla \varphi^\varepsilon|^2 dx = \int R_4^\varepsilon \cdot \varphi^\varepsilon dx. \quad (5.20)$$

In view of the Hölder inequality, one arrives at

$$\begin{aligned} \int R_4^\varepsilon \cdot \varphi^\varepsilon dx &\leq \|u\|_{L^\infty} \|\varphi^\varepsilon\|_{L^2} \|\nabla \varphi^\varepsilon\|_{L^2} + \|\nabla d^\varepsilon\|_{L^\infty} (\|v^\varepsilon\|_{L^2} \|\varphi^\varepsilon\|_{L^2} + \|\varphi^\varepsilon\|_{L^2} \|\nabla \varphi^\varepsilon\|_{L^2}) \\ &\quad + \|\nabla d\|_{L^\infty} \|\nabla \varphi^\varepsilon\|_{L^2} \|\varphi^\varepsilon\|_{L^2} + \|\nabla d\|_{L^\infty}^2 \|\varphi^\varepsilon\|_{L^2}^2 \\ &\leq C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{L^2}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2), \end{aligned}$$

which, together with (5.20), yields directly

$$\frac{1}{2} \frac{d}{dt} \int |\varphi^\varepsilon|^2 dx + \int |\nabla \varphi^\varepsilon|^2 dx \leq C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{L^2}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2). \quad (5.21)$$

Then the combination of (5.15), (5.19) and (5.21) yields immediately

$$\begin{aligned} &\frac{d}{dt} \int \left(\frac{p'(\rho)}{\rho} |\phi^\varepsilon|^2 + \frac{\rho}{2} |v^\varepsilon|^2 + \frac{1}{2} |\varphi^\varepsilon|^2 + \frac{1}{2} |\nabla \varphi^\varepsilon|^2 \right) dx + \mu \varepsilon \|v^\varepsilon\|_{H^1}^2 + \frac{3}{4} \int (|\nabla \varphi^\varepsilon|^2 + |\Delta \varphi^\varepsilon|^2) dx \\ &\leq C(\|\phi^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{L^2}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2) + C\varepsilon^{\frac{3}{2}}. \end{aligned}$$

which, together with the Grönwall inequality, completes the proof of Lemma 5.1. \square

Lemma 5.2. *For $t \in [0, \min\{T_0, T_1\}]$, it holds that*

$$\sup_{0 \leq \tau \leq t} \|\Delta \varphi^\varepsilon(\tau)\|_{L^2}^2 + \int_0^t \|\nabla \Delta \varphi^\varepsilon\|_{L^2}^2 d\tau \leq C\varepsilon^{\frac{1}{2}}. \quad (5.22)$$

Proof. Taking ∇ operator to (5.1)₃, we find

$$\nabla \varphi^\varepsilon - \nabla \Delta \varphi^\varepsilon = \nabla R_4^\varepsilon,$$

which, multiplying by $-\nabla \varphi^\varepsilon$, reads

$$-\int \partial_t \nabla \varphi^\varepsilon \cdot \nabla \Delta \varphi^\varepsilon dx + \int |\nabla \Delta \varphi^\varepsilon|^2 dx = -\int \nabla R_4^\varepsilon \cdot \nabla \Delta \varphi^\varepsilon dx. \quad (5.23)$$

Integrating by part and applying the boundary condition (5.2), it is easy to deduce

$$-\int \partial_t \nabla \varphi^\varepsilon \cdot \nabla \Delta \varphi^\varepsilon dx = -\int_{\partial\Omega} n \cdot \nabla \varphi^\varepsilon \cdot \nabla \Delta \varphi^\varepsilon d\sigma + \frac{1}{2} \frac{d}{dt} \int |\Delta \varphi^\varepsilon|^2 dx = \frac{1}{2} \frac{d}{dt} \int |\Delta \varphi^\varepsilon|^2 dx. \quad (5.24)$$

On the other hand, it is easy to check that

$$\begin{aligned} \|\nabla R_4^\varepsilon\|_{L^2}^2 &\leq C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{H^1}^2) + C(\|\nabla^2 \varphi^\varepsilon\|_{L^2}^2 + \|\nabla v^\varepsilon\|_{L^2}^2) \\ &\leq C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{H^1}^2) + C(\|\Delta \varphi^\varepsilon\|_{L^2}^2 + \|\nabla v^\varepsilon\|_{L^2}^2), \end{aligned} \quad (5.25)$$

where we have used the standard elliptic estimates in the last inequality. Hence, by virtue of the Cauchy inequality, (5.24) and (5.25), we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int |\Delta \varphi^\varepsilon|^2 dx + \int |\nabla \Delta \varphi^\varepsilon|^2 dx \\ &\leq \delta \|\nabla \Delta \varphi^\varepsilon\|_{L^2}^2 + C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{H^1}^2) + C(\|\Delta \varphi^\varepsilon\|_{L^2}^2 + \|\nabla v^\varepsilon\|_{L^2}^2). \end{aligned} \quad (5.26)$$

Choosing δ small enough in (5.26) and integrating over $[0, t]$, one attains

$$\begin{aligned} & \int |\Delta \varphi^\varepsilon(t)|^2 dx + \int_0^t \|\nabla \Delta \varphi^\varepsilon\|_{L^2}^2 d\tau \\ & \leq C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{H^1}^2) + C \int_0^t (\|\Delta \varphi^\varepsilon\|_{L^2}^2 + \|\nabla v^\varepsilon\|_{L^2}^2) d\tau \leq C\varepsilon^{\frac{1}{2}}, \end{aligned}$$

where we have used the estimate (5.3) in the last inequality. Therefore, we complete the proof of Lemma 5.2. \square

Lemma 5.3. *For $t \in [0, \min\{T_0, T_1\}]$, it holds that*

$$\begin{aligned} & \|(\operatorname{div} v^\varepsilon, \nabla(p^\varepsilon - p))\|_{L^2}^2 + (2\mu + \lambda)\varepsilon \int_0^t \|\nabla \operatorname{div} v^\varepsilon(\tau)\|_{L^2}^2 d\tau \\ & \leq \delta \int_0^t \|v_t^\varepsilon\|_{L^2}^2 d\tau + C_\delta \int_0^t \|(\varphi^\varepsilon, v^\varepsilon)\|_{H^1}^2 d\tau + C_\delta \varepsilon^{\frac{1}{2}}. \end{aligned} \quad (5.27)$$

Proof. Multiplying (5.1)₂ by $\nabla \operatorname{div} v^\varepsilon$, it is easy to deduce that

$$\begin{aligned} & \underbrace{\int (\rho v_t^\varepsilon + \rho u \cdot \nabla v^\varepsilon) dx}_{VIII_1} + \underbrace{\int \nabla(p^\varepsilon - p) \cdot \nabla \operatorname{div} v^\varepsilon dx}_{VIII_2} + \underbrace{\int \Phi^\varepsilon \cdot \nabla \operatorname{div} v^\varepsilon dx}_{VIII_3} \\ & = -\underbrace{\mu\varepsilon \int \nabla \times (\nabla \times v^\varepsilon) \cdot \nabla \operatorname{div} v^\varepsilon dx}_{VIII_4} + \underbrace{(2\mu + \lambda)\varepsilon \int |\nabla \operatorname{div} v^\varepsilon|^2 dx}_{VIII_5} \\ & \quad + \underbrace{\int R_2^\varepsilon \cdot \nabla \operatorname{div} v^\varepsilon dx}_{VIII_6} + \underbrace{\int R_3^\varepsilon \cdot \nabla \operatorname{div} v^\varepsilon dx}_{VIII_7}. \end{aligned} \quad (5.28)$$

Following the same argument as Lemma 6.2 of [12], it is easy to obtain the following estimates

$$\begin{aligned} VIII_1 & \leq -\frac{d}{dt} \int \frac{\rho}{2} |\operatorname{div} v^\varepsilon|^2 dx + \delta \|v_t^\varepsilon\|_{L^2}^2 + C_\delta \|\nabla v^\varepsilon\|_{L^2}^2 + C|v^\varepsilon|_{L^2(\partial\Omega)}, \\ VIII_2 & \leq -\frac{d}{dt} \int \frac{1}{2\gamma p^\varepsilon} |\nabla(p^\varepsilon - p)|^2 dx + C(1 + \|(u^\varepsilon, p^\varepsilon)\|_{W^{1,\infty}}) \|(p^\varepsilon - p, v^\varepsilon)\|_{H^1}^2, \\ VIII_3 & \leq C(1 + \|(\rho^\varepsilon, u^\varepsilon)\|_{W^{1,\infty}}) (\|(\varphi^\varepsilon, v^\varepsilon)\|_{H^1}^2 + \|(\varphi^\varepsilon, v^\varepsilon)|_{L^2(\partial\Omega)}), \\ VIII_4 & \leq \delta \varepsilon \|\nabla \operatorname{div} v^\varepsilon\|_{L^2}^2 + C_\delta \varepsilon (1 + \|v^\varepsilon\|_{H^1}^2), \\ VIII_6 & \leq \frac{(2\mu + \lambda)\varepsilon}{8} \|\nabla \operatorname{div} v^\varepsilon\|_{L^2}^2 + \delta \|v_t^\varepsilon\|_{L^2}^2 + C_\delta (\|(\varphi^\varepsilon, v^\varepsilon)\|_{H^1}^2 + \varepsilon^{\frac{3}{2}}). \end{aligned} \quad (5.29)$$

On the other hand, integrating by parts and applying the Cauchy inequality, one arrives at directly

$$\begin{aligned} VIII_7 & = - \int_{\partial\Omega} n \cdot (\nabla d^\varepsilon \cdot \Delta \varphi^\varepsilon + \nabla \varphi^\varepsilon \cdot \Delta d) \operatorname{div} v^\varepsilon d\sigma \\ & \quad + \int \operatorname{div}(\nabla d^\varepsilon \cdot \Delta \varphi^\varepsilon + \nabla \varphi^\varepsilon \cdot \Delta d) \operatorname{div} v^\varepsilon dx \\ & = \int \operatorname{div}(\nabla d^\varepsilon \cdot \Delta \varphi^\varepsilon + \nabla \varphi^\varepsilon \cdot \Delta d) \operatorname{div} v^\varepsilon dx \\ & \leq C(1 + \|(\nabla d^\varepsilon, \Delta d^\varepsilon)\|_{L^\infty}) (\|\nabla(\varphi^\varepsilon, v^\varepsilon)\|_{L^2}^2 + \|\Delta \varphi^\varepsilon\|_{L^2}^2 + \|\nabla \Delta \varphi^\varepsilon\|_{L^2}^2). \end{aligned} \quad (5.30)$$

Substituting (5.29) and (5.30) into (5.28), we find

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{\rho}{2} |\operatorname{div} v^\varepsilon|^2 + \frac{1}{2\gamma p^\varepsilon} |\nabla(p^\varepsilon - p)|^2 \right) + (2\mu + \lambda)\varepsilon \int |\nabla \operatorname{div} v^\varepsilon|^2 dx \\ & \leq \delta \|v_t^\varepsilon\|_{L^2}^2 + C\|(\varphi^\varepsilon, v^\varepsilon)\|_{H^1}^2 + C\|(\varphi^\varepsilon, v^\varepsilon)|_{L^2(\partial\Omega)} + C\varepsilon^{\frac{3}{2}} \\ & \quad + C(\|\nabla \varphi^\varepsilon\|_{L^2}^2 + \|\Delta \varphi^\varepsilon\|_{L^2}^2 + \|\nabla \Delta \varphi^\varepsilon\|_{L^2}^2). \end{aligned} \quad (5.31)$$

By virtue of the trace theorem in Proposition 2.3, we obtain

$$|(\varphi^\varepsilon, v^\varepsilon)|_{L^2} \leq C(\|(\varphi^\varepsilon, v^\varepsilon)\|_{H^1} + \|(\varphi^\varepsilon, v^\varepsilon)\|_{L^2}^{\frac{2}{3}}) \leq (\|(\varphi^\varepsilon, v^\varepsilon)\|_{H^1} + \varepsilon^{\frac{1}{2}}). \quad (5.32)$$

Integrating (5.31) over $[0, t]$ and substituting (5.32) into the resulting inequality, we find

$$\begin{aligned} & \|(\operatorname{div} v^\varepsilon, \nabla(p^\varepsilon - p))\|_{L^2}^2 + (2\mu + \lambda)\varepsilon \int_0^t \|\nabla \operatorname{div} v^\varepsilon(\tau)\|_{L^2}^2 d\tau \\ & \leq \delta \int_0^t \|v_t^\varepsilon\|_{L^2}^2 d\tau + C_\delta \int_0^t \|(\phi^\varepsilon, v^\varepsilon)\|_{H^1}^2 d\tau + C_\delta \varepsilon^{\frac{1}{2}}. \end{aligned}$$

Therefore, we complete the proof of Lemma 5.3. \square

Lemma 5.4. *For $t \in [0, \min\{T_0, T_1\}]$, it holds that*

$$\begin{aligned} & \|\nabla \times v^\varepsilon\|_{L^2}^2 + \varepsilon \int_0^t \|(\nabla \times v^\varepsilon)(\tau)\|_{H^1}^2 d\tau \\ & \leq \delta \|\nabla(\phi^\varepsilon, v^\varepsilon)\|_{L^2}^2 + C_\delta \int_0^t (\|v_t^\varepsilon\|_{L^2}^2 + \varepsilon \|\nabla^2 v^\varepsilon\|_{L^2}^2) d\tau + C_\delta \int_0^t \|(\phi^\varepsilon, v^\varepsilon)\|_{H^1}^2 d\tau + C_\delta \varepsilon^{\frac{1}{6}}. \end{aligned} \quad (5.33)$$

Proof. Multiplying by (5.1)₂ by $\nabla \times (\nabla \times v^\varepsilon)$ yields immediately

$$\begin{aligned} & \underbrace{\int \rho^\varepsilon v_t^\varepsilon \cdot \nabla \times (\nabla \times v^\varepsilon) dx}_{IX_1} + \underbrace{\int \nabla(p^\varepsilon - p) \cdot \nabla \times (\nabla \times v^\varepsilon) dx}_{IX_2} \\ & = -\mu \varepsilon \|\nabla \times (\nabla \times v^\varepsilon)\|_{L^2}^2 + (2\mu + \lambda)\varepsilon \int \nabla \operatorname{div} v^\varepsilon \cdot \nabla \times (\nabla \times v^\varepsilon) dx \\ & \quad - \underbrace{\int \tilde{\Phi}^\varepsilon \cdot \nabla \times (\nabla \times v^\varepsilon) dx}_{IX_3} + \underbrace{\int \tilde{R}_2^\varepsilon \cdot \nabla \times (\nabla \times v^\varepsilon) dx}_{IX_4} + \underbrace{\int R_3^\varepsilon \cdot \nabla \times (\nabla \times v^\varepsilon) dx}_{IX_5}, \end{aligned} \quad (5.34)$$

where

$$\begin{aligned} \tilde{\Phi}^\varepsilon &= \rho^\varepsilon u^\varepsilon \cdot \nabla v^\varepsilon + (\rho^\varepsilon u^\varepsilon - \rho u) \cdot \nabla u, \\ \tilde{R}_2^\varepsilon &= -\phi^\varepsilon u_t + \mu \varepsilon \Delta u + (\mu + \lambda)\varepsilon \nabla \operatorname{div} u. \end{aligned}$$

Following the same argument as Lemma 6.3 of [12], it is easy to obtain the following estimates

$$\begin{aligned} IX_1 & \geq \frac{d}{dt} \left\{ \int \frac{\rho^\varepsilon}{2} |\nabla \times v^\varepsilon|^2 dx + \int_{\partial\Omega} \left(\frac{\rho^\varepsilon}{2} v^\varepsilon (Bv^\varepsilon) + \rho^\varepsilon v^\varepsilon \cdot (Bu - n \times w) \right) d\sigma \right\} \\ & \quad - \delta \|v_t^\varepsilon\|_{L^2}^2 - C_\delta (\|v^\varepsilon\|_{H^1}^2 + |v^\varepsilon|_{L^2}), \\ |IX_2| & \leq C(\|p^\varepsilon - p\|_{H^1}^2 + \|v^\varepsilon\|_{H^1}^2 + |p^\varepsilon - p|_{L^2}), \\ |IX_3| & \leq C(\|(\phi^\varepsilon, v^\varepsilon)\|_{H^1}^2 + |(\phi^\varepsilon, v^\varepsilon)|_{L^2}), \\ |IX_4| & \leq \delta \varepsilon \|\nabla \times (\nabla \times v^\varepsilon)\|_{L^2}^2 + C_\delta (\|(\phi^\varepsilon, v^\varepsilon)\|_{H^1}^2 + |(\phi^\varepsilon, v^\varepsilon)|_{L^2} + \varepsilon^{\frac{3}{2}}). \end{aligned} \quad (5.35)$$

On the other hand, integrating by part and applying the boundary condition (5.2), we find

$$\begin{aligned} IX_5 &= \int_{\partial\Omega} R_3^\varepsilon \cdot (n \times (\nabla \times v^\varepsilon)) d\sigma + \int \nabla \times R_3^\varepsilon \cdot \nabla \times v^\varepsilon dx \\ &= \int_{\partial\Omega} R_3^\varepsilon \cdot [Bv^\varepsilon]_\tau d\sigma + \int_{\partial\Omega} R_3^\varepsilon \cdot ([Bu]_\tau - n \times w) d\sigma \\ & \quad + \int \nabla \times R_3^\varepsilon \cdot \nabla \times v^\varepsilon dx \\ &= IX_{51} + IX_{52} + IX_{53}. \end{aligned} \quad (5.36)$$

Integrating by part and applying the Hölder inequality, we obtain

$$\begin{aligned}
IX_{51} &= \int_{\partial\Omega} (n \times R_3^\varepsilon) \cdot (n \times [Bv^\varepsilon]_\tau) d\sigma \\
&= \int_{\partial\Omega} (n \times R_3^\varepsilon) \cdot (n \times (Bv^\varepsilon)) d\sigma \\
&= \int (\nabla \times R_3^\varepsilon) \cdot (n \times (Bv^\varepsilon)) dx + \int R_3^\varepsilon \cdot \nabla \times (n \times (Bv^\varepsilon)) d\sigma \\
&\leq C(\|v^\varepsilon\|_{H^1}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2 + \|\nabla \Delta \varphi^\varepsilon\|_{L^2}^2).
\end{aligned} \tag{5.37}$$

It is easy to check that

$$|IX_{52}| \leq C(\|\nabla \varphi^\varepsilon\|_{L^2(\partial\Omega)} + |\Delta \varphi^\varepsilon|_{L^2(\partial\Omega)}), \tag{5.38}$$

and

$$|IX_{53}| \leq C(\|\nabla \times v^\varepsilon\|_{L^2}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2 + \|\nabla \Delta \varphi^\varepsilon\|_{L^2}^2). \tag{5.39}$$

Then, substituting (5.37)-(5.39) into (5.36) yields

$$IX_5 \leq C(\|v^\varepsilon\|_{H^1}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2 + \|\nabla \Delta \varphi^\varepsilon\|_{L^2}^2) + C(\|\nabla \varphi^\varepsilon\|_{L^2(\partial\Omega)} + |\Delta \varphi^\varepsilon|_{L^2(\partial\Omega)}). \tag{5.40}$$

The application of the trace theorem in Proposition 2.3 yields that

$$\begin{aligned}
|(\phi^\varepsilon, v^\varepsilon)|_{L^2} &\leq C\|(\phi^\varepsilon, v^\varepsilon)\|_{H^1}^{\frac{1}{2}} \|(\phi^\varepsilon, v^\varepsilon)\|_{L^2}^{\frac{1}{2}} \leq C\|(\phi^\varepsilon, v^\varepsilon)\|_{H^1}^2 + C\varepsilon^{\frac{1}{2}}, \\
\|\nabla \varphi^\varepsilon\|_{L^2(\partial\Omega)} &\leq C\|\nabla \varphi^\varepsilon\|_{H^1} \leq C\varepsilon^{\frac{1}{2}}, \\
|\Delta \varphi^\varepsilon|_{L^2(\partial\Omega)} &\leq \|\Delta \varphi^\varepsilon\|_{H^1}^{\frac{1}{2}} \|\Delta \varphi^\varepsilon\|_{L^2}^{\frac{1}{2}} \leq C\|\nabla \Delta \varphi^\varepsilon\|_{L^2}^2 + C\varepsilon^{\frac{1}{6}}.
\end{aligned} \tag{5.41}$$

Substituting (5.35), (5.40) and (5.41) into (5.34) reads immediately

$$\begin{aligned}
&\frac{d}{dt} \left\{ \int \frac{\rho^\varepsilon}{2} |\nabla \times v^\varepsilon|^2 dx + \int_{\partial\Omega} \left(\frac{\rho^\varepsilon}{2} v^\varepsilon (Bv^\varepsilon) + \rho^\varepsilon v^\varepsilon \cdot (Bu - n \times w) \right) d\sigma \right\} \\
&\quad + \frac{\mu\varepsilon}{2} \int |\nabla \times (\nabla \times v^\varepsilon)|^2 dx \\
&\leq C\delta \|v_t^\varepsilon\|_{L^2}^2 + C\delta\varepsilon \|\nabla^2 v^\varepsilon\|_{L^2}^2 + C_\delta (\|(\phi^\varepsilon, v^\varepsilon)\|_{H^1}^2 + \varepsilon^{\frac{1}{6}}).
\end{aligned} \tag{5.42}$$

In view of the Proposition 2.1, one arrives at

$$\begin{aligned}
\|\nabla \times v^\varepsilon\|_{H^1}^2 &\leq C_1(\|\nabla \times (\nabla \times v^\varepsilon)\|_{L^2}^2 + \|\operatorname{div}(\nabla \times v^\varepsilon)\|_{L^2}^2 + \|\nabla \times v^\varepsilon\|_{L^2}^2 + |n \times (\nabla \times v^\varepsilon)|_{H^{\frac{1}{2}}}^2) \\
&\leq C_1(\|\nabla \times (\nabla \times v^\varepsilon)\|_{L^2}^2 + \|\nabla \times v^\varepsilon\|_{L^2}^2 + |Bv^\varepsilon|_{H^{\frac{1}{2}}}^2 + |(Bu)_\tau - n \times w|_{H^{\frac{1}{2}}}^2).
\end{aligned} \tag{5.43}$$

By virtue of the trace inequality in Proposition 2.3, we have

$$\|(\phi^\varepsilon, v^\varepsilon)\|_{L^2}^2 \leq C\|(\phi^\varepsilon, v^\varepsilon)\|_{H^1} \|(\phi^\varepsilon, v^\varepsilon)\|_{L^2} \leq \delta \|\nabla(\phi^\varepsilon, v^\varepsilon)\|_{L^2}^2 + C_\delta \varepsilon^{\frac{3}{2}}. \tag{5.44}$$

Substituting (5.43) and (5.44) into (5.42) and integrating the resulting inequality over $[0, t]$ yield the estimate (5.33). Therefore, we complete the proof of Lemma 5.4. \square

Proof for Theorem 1.3: By virtue of Proposition 2.1, we have

$$\begin{aligned}
\|v^\varepsilon\|_{H^1}^2 &\leq C(\|\nabla \times v^\varepsilon\|_{L^2}^2 + \|\operatorname{div} v^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2 + \|v^\varepsilon \cdot n\|_{H^{\frac{1}{2}}}^2) \\
&\leq C(\|\nabla \times v^\varepsilon\|_{L^2}^2 + \|\operatorname{div} v^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2),
\end{aligned} \tag{5.45}$$

and

$$\begin{aligned}
\|v^\varepsilon\|_{H^2}^2 &\leq C(\|\nabla \times v^\varepsilon\|_{L^2}^2 + \|\operatorname{div} v^\varepsilon\|_{H^1}^2 + \|v^\varepsilon\|_{H^1}^2 + \|v^\varepsilon \cdot n\|_{H^{\frac{3}{2}}}^2) \\
&\leq C(\|\nabla \times v^\varepsilon\|_{H^2}^2 + \|\operatorname{div} v^\varepsilon\|_{H^1}^2 + \|v^\varepsilon\|_{H^1}^2).
\end{aligned} \tag{5.46}$$

On the other hand, it follows from the equation (5.1)₂ that

$$\|v_t^\varepsilon\|_{L^2}^2 \leq C(\|(\phi^\varepsilon, v^\varepsilon)\|_{H^1}^2 + \varepsilon^2 \|\nabla^2 v^\varepsilon\|_{L^2}^2 + \varepsilon^{\frac{1}{2}}). \tag{5.47}$$

The combination of (5.27), (5.33), (5.45)-(5.47) and choosing δ small enough, one obtains that

$$\|\nabla(v^\varepsilon, p^\varepsilon - p)\|_{L^2}^2 + \varepsilon \int_0^t \|v^\varepsilon(\tau)\|_{H^2}^2 d\tau \leq C \int_0^t \|\nabla(v^\varepsilon, p^\varepsilon - p)\|_{L^2}^2 d\tau + C\varepsilon^{\frac{1}{6}},$$

which, together with the Grönwall inequality, gives

$$\|\nabla(v^\varepsilon, p^\varepsilon - p)\|_{L^2}^2 + \varepsilon \int_0^t \|v^\varepsilon(\tau)\|_{H^2}^2 d\tau \leq C\varepsilon^{\frac{1}{6}}. \quad (5.48)$$

On the other hand, by virtue of Sobolev inequality, uniform estimate (1.19) and convergence rate (5.3), it is easy to deduce

$$\|(\rho^\varepsilon - \rho, u^\varepsilon - u)\|_{L^\infty(0, T_2; L^\infty(\Omega))} \leq C\|(\rho^\varepsilon - \rho, u^\varepsilon - u)\|_{L^2}^{\frac{2}{5}} \|(\rho^\varepsilon - \rho, u^\varepsilon - u)\|_{W^{1, \infty}}^{\frac{3}{5}} \leq C\varepsilon^{\frac{3}{10}}, \quad (5.49)$$

and

$$\|d^\varepsilon - d\|_{L^\infty(0, T^2; W^{1, \infty}(\Omega))} \leq C\|d^\varepsilon - d\|_{H^1}^{\frac{2}{5}} \|d^\varepsilon - d\|_{W^{2, \infty}}^{\frac{3}{5}} \leq C\varepsilon^{\frac{3}{10}}, \quad (5.50)$$

The combination of (5.3), (5.22) and (5.48)-(5.50) completes the proof of Theorem 1.3

ACKNOWLEDGEMENTS

The author Jincheng Gao would like to thank Yong Wang for fruitful discussion and suggestion.

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